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Diffraction Theory - Three Papers

Translated from the Russian by
ABE SHENITZER

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DIFFRACTION THEORY - THREE PAPERS

Translated from the Russian by Abe Shenitzer

Abe Shenitzer

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Morris Kline

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Project Director

March, 1957

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The following three papers were translated by us because these Russian efforts are related to our own researches in diffraction theory. It was decided to bring out the translations as a report to make them more generally available to scientists working in this field.

Table of Contents

Page

The Electromagnetic Field of a Line Source Located in the Interior of an Ideally Conducting Parabolic Screen

- G.A. Grinberg, N.N. Lebedev, I.P. Skalskaia and Ya.S. Uflyand

1. Formulation of the problem 1

2. Construction of a formal solution of the problem 3

3. Investigation of the integral representing the secondary field and proof of the regularity of the obtained solution 7

4. Various representations of the solution 10

5. Test of the formal solution 14

6. The uniqueness theorem 16

7. Investigation of the solution for large values of the wave number k. Transition to geometrical optics 19

8. Generalization of the solution to the case of arbitrary location of the source on the axis of the cylinder 22

Supplements 24

References 27

The Electromagnetic Field of a Dipole Located in the Interior of a Parabolic Reflector

- I. P. Skalskaya

1. Formulation of the problem and representation of the solution by means of complex integrals 1

2. Determination of the coefficients of the representations 6

3. Investigation of the solution. Transition to geometrical optics. 11

References 18

Appendix - W. Magnus and N.N. Lebedev 19

Table of Contents (continued)

Page

Derivation of the Laws of Reflection of Geometrical Optics on the Basis of an Asymptotic Treatment of the Diffraction Problem

-M.I. Kontorovitch and U.K. Muravev; Foreword by M. Kline

1. Preliminary remarks 1

2. Derivation of the asymptotic formula 2

3. Derivation of asymptotic formulas for the computation of the electromagnetic field reflected from a surface of arbitrary shape 13

Bibliography 22

Figures 23

THE ELECTROMAGNETIC FIELD OF A LINE SOURCE LOCATED IN THE INTERIOR OF AN IDEALLY CONDUCTING PARABOLIC SCREEN

G.A. Grinberg, N.N. Lebedev, I.P. Skalskaia and Ya.S. Uflyand

(Translated by Abe Shenitzer)

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Abstract

On the basis of a critical survey of earlier papers it is shown that the results obtained in these papers are insufficiently justified. A rigorous solution of the problem is given. It is also shown that in the limiting case of very high frequencies the solution takes on a form which corresponds to the approximation yielded by geometrical optics.

At first sight the problem at hand appears to be almost trivial, since the variables in the fundamental equation can be separated. However, it is not at all simple to choose special solutions which can be used to construct a solution of the problem satisfying all requirements including the radiation condition at infinity and analyticity at the focus of the cylinder; nor is it a simple matter to construct the solution itself. In particular, to the best of our knowledge, the papers [1-4] devoted to the wave equation for a parabolic domain do not contain a suitably justified solution of the problem at hand[*]. In the present paper this gap has been filled. It is also shown that in the limiting case of very high frequencies the solution takes on a form which corresponds to the approximation yielded by geometrical optics.

1. Formulation of the problem

We consider the problem of reflection of electromagnetic waves from a conducting screen which has the form of a parabolic cylinder. We assume that the source of vibrations is a line source located in the interior of the cylinder on the focal line[**] of the cylinder and that the intensity of the current in the source is $I = I_o e^{i\omega t}$, where I, the amplitude of the current, is constant and ω is the circular frequency.

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[*] For details see Chapter 1 of our paper [5].

[**] The case of an arbitrary location of the source is considered in section 8.

We choose a system of Cartesian coordinates (x,y,z) so that the x-axis lies in the plane of symmetry of the parabolic cylinder and the z-axis coincides with the current. In addition to the Cartesian coordinates we shall use parabolic coordinates (α,β) defined by the relation

(1.1) $$x + iy = 1/2(\alpha + i\beta)^2, \qquad -\infty < \alpha < +\infty, \quad 0 \leq \beta < \infty,$$

which implies that

(1.2) $$x = 1/2(\alpha^2 - \beta^2), \qquad y = \alpha\beta.$$

The equation of the surface of the cylinder is then seen to be $\beta = \beta_0$ (Fig. 1), where β_0 is a constant.

In the case under consideration the electric field vector E has only one component E_z of the form

(1.3) $$E_z = E(x,y)e^{i\omega t} \equiv Ee^{i\omega t} .$$

Separating out the field of the source we put

(1.4) $$E = E_0 + u,$$

where

(1.5) $$E_0 = -\pi\omega Ic^{-2} H_0^{(2)}(kr); \quad r = \sqrt{x^2+y^2} = 1/2(\alpha^2+\beta^2),$$

where $k = \omega/c$, $c =$ the velocity of light in empty space, $H_0^{(2)}(z)$ is a Hankel function of the second kind and u is the secondary electric field.

Since the electrical field E must satisfy the Helmholtz equation $\Delta E + k^2 E = 0$ as well as the radiation condition and since it must vanish on the surface of the conducting screen, we can give the following mathematical formulation of the problem under consideration: find a solution $E(\alpha,\beta)$ of the equation

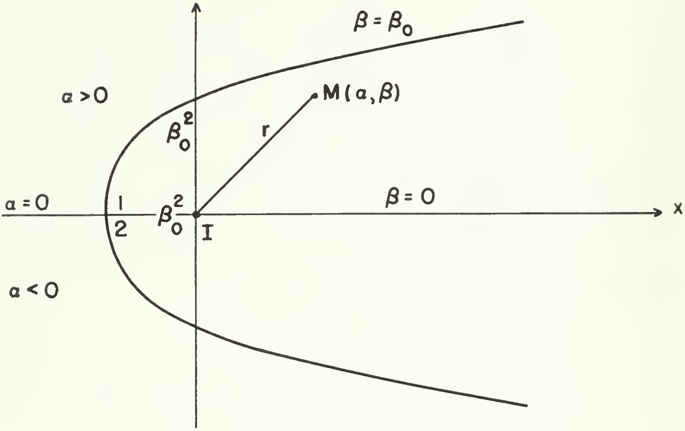
$\beta = \beta_0$
$\alpha > 0$
$M(\alpha,\beta)$
r
β_0^2
$\alpha = 0$
$\frac{1}{2}\beta_0^2$
$\beta = 0$
x
I
$\alpha < 0$

Figure 1

$$(1.6) \qquad (\partial^2 E/\partial\alpha^2) + (\partial^2 E/\partial\beta^2) + k^2(\alpha^2 + \beta^2)E = 0.$$

which is regular in the interior of the parabolic cylinder [with the exception of the line $\alpha = \beta = 0$ in the vicinity of which

$$(1.7) \qquad E = -\frac{\pi\omega I}{c^2} H_o^{(2)}\left(k\frac{\alpha^2+\beta^2}{2}\right) + u$$

(where u is a regular function)], and satisfies the boundary condition

$$(1.8) \qquad E\Big|_{\beta = \beta_o} = 0$$

and the radiation condition*

$$(1.9) \qquad E\Big|_{|\alpha|\to\infty} = O(|\alpha|^{-1/2}); \left(\frac{1}{\alpha}\frac{\partial E}{\partial\alpha} + ikE\right)_{|\alpha|\to\infty} = O(|\alpha|^{-1/2})$$

(uniformly in β).

In the sequel (cf. section 6) we shall show that in this formulation the problem has a unique solution.

2. Construction of a formal solution of the problem

To achieve separation of variables in the wave equation

$$(2.1) \qquad (\partial^2 u/\partial\alpha^2) + (\partial^2 u/\partial\beta^2) + k^2(\alpha^2 + \beta^2)u = 0$$

we put

$$(2.2) \qquad u = A(\alpha)B(\beta).$$

Then we obtain for the factors A and B the ordinary differential equations

---

*The relations (1.9) follow in the usual manner from the condition that in a unit of time the same amount of energy radiates to the outside across every section α = const. of the guide and that no waves are reflected back from infinity.

$$(2.3) \qquad A'' + (k^2\alpha^2 + \lambda)A = 0; \qquad B'' + (k^2\beta^2 - \lambda)B = 0,$$

where λ is the separation parameter.

In the first of these equations we make use of the substitution

$$(2.4) \qquad A(\alpha) = \exp\left\{- ik\alpha^2/2\right\} v(\xi); \qquad \xi = \sqrt{ik}\,\alpha,$$

and we put

$$(2.5) \qquad \lambda = ik(2\nu + 1),$$

this yields the following equation for the function $v(\xi)$

$$(2.6) \qquad v'' - 2\xi v' + 2\nu v = 0.$$

The general integral of equation (2.6) is

$$(2.7) \qquad v(\xi) = MF\left(-\frac{\nu}{2}, \frac{1}{2}, \xi^2\right) + N\xi F\left(\frac{1-\nu}{2}, \frac{3}{2}, \xi^2\right),$$

where M and N are constants and

$$(2.8) \qquad F(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{z^n}{n!}$$

is the confluent hypergeometric function.

Since in our case the solution must be an even function of the variable α we must have $N = 0$, so that*

$$(2.9) \qquad A(\alpha) = A_\nu^{(1)}(\alpha) = \exp\left\{- ik\alpha^2/2\right\} F(-\nu/2, 1/2, ik\alpha^2)$$
$$= (ik\alpha^2)^{-1/4} M_{\nu/2+1/4}(ik\alpha^2).$$

Here we may restrict ourselves to the interval $0 \leq \alpha < \infty$ which is what is done in the sequel.

---

* cf., for instance, [6,7].

If we make use of the asymptotic representation for the function $A_\nu^{(1)}(\alpha)$ for $\alpha \to \infty$ [*], namely

$$(2.10) \qquad A_\nu^{(1)}(\alpha)\Big|_{\alpha\to\infty} \approx \emptyset(\nu,\alpha)\exp\left\{ik\alpha^2/2\right\}\alpha^{-(\nu+1)} + \Psi(\nu,\alpha)\exp\left\{-ik\alpha^2/2\right\}\alpha^{\nu}$$

where $\emptyset$ and Ψ are functions which tend to finite limits as $\alpha \to \infty$ and if we require that the special solutions which we are seeking be in the nature of waves going out to infinity with amplitudes tending to zero, then we obtain for the parameter ν the condition

$$(2.11) \qquad -1/2 < \mathrm{Re}\left\{\nu\right\} < 0.$$

For a chosen value of λ the general integral of the second of the two equations (2.3) is

$$(2.12) \qquad B = B_\nu(\beta) = \exp\left\{-ik\beta^2/2\right\}\left[PF\left(\frac{1+\nu}{2}, \frac{1}{2}, ik\beta^2\right) + Q\sqrt{ik}\,\beta F\left(1+\frac{\nu}{2}, \frac{3}{2}, ik\beta^2\right)\right],$$

where P and Q are constants.

In view of the boundedness of grad u at the focus it follows that $Q = 0$[**], so that

$$(2.13) \qquad B_\nu(\beta) \equiv B_\nu^{(1)}(\beta) = \exp\left\{-ik\beta^2/2\right\} F\left(\frac{1+\nu}{2}, \frac{1}{2}, ik\beta^2\right).$$

Thus we arrive at the special solutions of equation (2.1)

$$(2.14) \qquad u_\nu = A_\nu^{(1)}(\alpha)B_\nu^{(1)}(\beta); \quad -1/2 < R\left\{\nu\right\} < 0,$$

and a possible solution of our problem is given by the complex integral

---

[*] See Supplement 1, formula (D,1) as well as [8]

[**] This result is obtained by substituting the special solutions under consideration in the equations (3.5) in the sequel.

$$(2.15) \qquad u(\alpha,\beta) = \int_{-\delta-i\infty}^{-\delta+i\infty} C(\nu)A_\nu^{(1)}(\alpha)B_\nu^{(1)}(\beta)d\nu; \qquad 0 < \delta < 1/2.$$

The unknown function $C(\nu)$ must be determined from the boundary condition (1.8), i.e., from the equality

$$(2.16) \quad \int_{-\delta-i\infty}^{-\delta+i\infty} C(\nu)A_\nu^{(1)}(\alpha)B_\nu^{(1)}(\beta_o)d\nu = \frac{\pi\omega I}{c^2} H_o^{(2)}\left(k\,\frac{\alpha^2+\beta_o^2}{2}\right) \quad (-\infty < \alpha < +\infty)$$

Thus the problem is that of representing the field of the source as an integral with respect to the functions $A_\nu^{(1)}(\alpha)$. Such a representation can be obtained by making use of Magnus' decomposition theorem[3]

$$(2.17) \qquad f(\alpha) = \int_{-\delta-i\infty}^{-\delta+i\infty} F(\nu)A_\nu^{(1)}(\alpha)d\nu;$$

$$F(\nu) = \frac{\sqrt{k}\;e^{-i\pi/4}}{2\pi^2}\,\Gamma(-\tfrac{\nu}{2})\,\Gamma(\tfrac{\nu+1}{2})e^{i\pi\nu/2}\int_0^\infty f(\alpha)A_\nu^{(1)}(\alpha)d\alpha \qquad (0 < \delta < 1),$$

from which it follows that

$$(2.18)\,C(\nu) = \frac{\omega\sqrt{k}\,I\,e^{-i\pi/4}\,\Gamma(-\frac{\nu}{2})\Gamma(\frac{\nu+1}{2})e^{i\pi\nu/2}}{2\pi c^2\,B_\nu^{(1)}(\beta_o)}\int_0^\infty H_o^{(2)}\left(k\,\frac{\alpha^2+\beta^2}{2}\right)A_\nu^{(1)}(\alpha)d\alpha.$$

Making use of the fact that

$$(2.19) \qquad \int_0^\infty H_o^{(2)}\left(k\,\frac{\alpha^2+\beta^2}{2}\right)A_\nu^{(1)}(\alpha)d\alpha = (\pi k)^{-1/2}\,e^{i\pi/4}\,B_\nu^{(3)}(\beta),$$

where*

---

*

$$B_\nu^{(3)}(\beta) = \Gamma(\tfrac{1}{2})\,\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\nu/2)}\,B_\nu^{(1)}(\beta) + \Gamma(-\tfrac{1}{2})B_\nu^{(2)}(\beta)$$

where

$$B_\nu^{(2)}(\beta) = \exp\left[-ik\beta^2/2\right]\sqrt{ik}\,\beta\,F(1+\tfrac{\nu}{2},\,\tfrac{3}{2},\,ik\beta^2)$$

is an odd solution of the second of the equations (2.3).

$$B_\nu^{(3)}(\beta) = (ik\beta^2)^{-1/4}\,\Gamma\left(\frac{\nu+1}{2}\right) W_{-\frac{1}{4}-\frac{\nu}{2},\,\frac{1}{4}}(ik\beta^2) \tag{2.20}$$

($W_{\lambda,\mu}(z)$ is Whittaker's function[9]) we obtain the final form of the solution of our problem

$$u(\alpha,\beta) = \frac{\omega T}{2\pi\sqrt{\pi}c^2}\int_{-\delta-i\infty}^{-\delta+i\infty}\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}\,\frac{B_\nu^{(3)}(\beta_o)}{B_\nu^{(1)}(\beta_o)}\,B_\nu^{(1)}(\beta)A_\nu^{(1)}(\alpha)\,d\nu \tag{2.21}$$

$$(0 < \delta < 1/2).$$

It is clear that this formal solution requires detailed verification. This verification is found in section 5.

3. Investigation of the integral representing the secondary field and proof of the regularity of the obtained solution

1. The functions which appear in the integrand in (2.21) are entire functions and as such are continuous functions of the variable ν. Since the function $B_\nu^{(1)}(\beta_o)$ does not vanish along the path of integration (see supplement 2), it follows that the integral exists for every choice of finite limits of integration.

To prove convergence of the integral for infinite limits of integration we shall make use of the asymptotic representations for the functions $A_\nu^{(1)}(\alpha)$, $B_\nu^{(1)}(\beta)$, and $B_\nu^{(3)}(\beta)$ for fixed α and β ($A < A$, $0 \leq \beta < \infty$), $|\nu| \to \infty$, $R\left\{\nu\right\} \geq -\delta > -\frac{1}{2}$ *. Putting $\nu = -\delta + i\tau$, we have for $\tau \to -\infty$:

$$|A_\nu^{(1)}(\alpha)| = O(1); \quad |B_\nu^{(1)}(\beta)| = O\left(\exp\left\{\sqrt{2k|\tau|}\,\beta\right\}\right); \tag{3.1}$$

$$|B_\nu^{(3)}(\beta)| = O\left(|\tau|^{-1/2}\exp\left\{-\sqrt{2k|\tau|}\,\beta\right\}\right).$$

Since

---

* cf. formulas (D,3), (D,4) and (D,5).

$$\left|\Gamma(-\tfrac{\nu}{2})\Gamma(\tfrac{\nu+1}{2})e^{i\pi\nu/2}\right| = O\left(|\tau|^{-1/2}\right),$$

it follows that

$$\left|\int_{-\delta-i\infty}^{-\delta-iT} \Gamma(-\tfrac{\nu}{2})\Gamma(\tfrac{\nu+1}{2})e^{i\pi\nu/2}\,\frac{B_\nu^{(3)}(\beta_o)}{B_\nu^{(1)}(\beta_o)}\,B_\nu^{(1)}(\beta)A_\nu^{(1)}(\alpha)d\nu\right| \leq$$

$$(3.2) \qquad \leq M\int_T^\infty \exp\left\{-\sqrt{2k|\tau|}\,\beta_o\right\}\frac{d|\tau|}{|\tau|},$$

which implies the uniform convergence of the integral (2.21) with respect to the lower integral.

In just the same way we find for $\tau \to +\infty$:

$$(3.3) \qquad |A_\nu^{(1)}(\alpha)| = O\left(\exp\left\{\sqrt{2k\tau}\,\alpha\right\}\right);$$

$$|B_\nu^{(1)}(\beta)| = O(\tau^{-1/2});$$

$$|B_\nu^{(3)}(\beta)| = O(\tfrac{1}{\sqrt{\tau}}); \quad \left|\Gamma(-\tfrac{\nu}{2})\Gamma(\tfrac{\nu+1}{2})e^{i\pi\nu/2}\right| = O(\tau^{-1/2}e^{-\pi\tau})$$

whence

$$\left|\int_{-\delta+iT}^{-\delta+i\infty} \Gamma(-\tfrac{\nu}{2})\Gamma(\tfrac{\nu+1}{2})e^{i\pi\nu/2}\,\frac{B_\nu^{(3)}(\beta_o)}{B_\nu^{(1)}(\beta_o)}\,B_\nu^{(1)}(\beta)A_\nu^{(1)}(\alpha)d\nu\right| \leq$$

$$(3.4) \qquad \leq M\int_T^\infty e^{-\pi\tau}\,e^{\sqrt{2k\tau}}\,\frac{d\tau}{\sqrt{\tau}}$$

so that the integral (2.21) converges uniformly also with respect to the upper limit.

The uniform convergence of the integral (2.21) implies that u is a continuous function of the variables α and β in the domain under consideration.

2. To prove analyticity of the function u we must still prove the existence and continuity of the first two derivatives of u with respect to x and y. For this it suffices to prove the uniform convergence of the integrals obtained after differentiation under the integral sign.

The estimates

$$|A_\nu^{(1)'}(\alpha)| = O(|\tau|); \quad |B_\nu^{(1)'}(\beta)| = O\left(|\tau| \exp\left\{\sqrt{2k|\tau|}\,\beta\right\}\right) \text{ as } \tau \to -\infty$$

imply the uniform convergence with respect to the lower limit of the integrals obtained from (2.21) by differentiating under the integral sign with respect to α and β. In exactly the same way, making use of the estimates

$$|A_\nu^{(1)'}(\alpha)| = O\left(\tau \exp\left\{\sqrt{2k\tau}\,\alpha\right\}\right); \quad |B_\nu^{(1)'}(\beta)| = O(\tau) \quad \text{as } \tau \to +\infty,$$

we can prove the uniform convergence of the integrals in question with respect to the upper limit. Analogous reasoning proves the existence and continuity of derivatives of any order of u with respect to α and β.

We now consider derivatives of u with respect to x and y. These derivatives are given by the relations

$$(3.5) \qquad \frac{\partial u}{\partial x} = \frac{\alpha}{\alpha^2+\beta^2}\frac{\partial u}{\partial \alpha} - \frac{\beta}{\alpha^2+\beta^2}\frac{\partial u}{\partial \beta}; \quad \frac{\partial u}{\partial y} = \frac{\alpha}{\alpha^2+\beta^2}\frac{\partial u}{\partial \beta} + \frac{\beta}{\alpha^2+\beta^2}\frac{\partial u}{\partial \alpha}.$$

These formulas show that $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in the domain under consideration with the possible exception of the point $\alpha = \beta = 0$. To investigate the behavior of $\partial u/\partial x$ and $\partial u/\partial y$ at $\alpha = \beta = 0$ we put

$$(3.6) \qquad \alpha = r\cos\theta; \quad \beta = r\sin\theta \quad (0 \le \theta \le \pi)$$

and consider the expression

$$(3.7) \qquad \frac{\partial u}{\partial x} = \frac{\omega I}{2\pi\sqrt{\pi}\,c^2}\int_{-\delta-i\infty}^{-\delta+i\infty} \Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2}\,\frac{B_\nu^{(3)}(\beta_0)}{B_\nu^{(1)}(\beta_0)}\,\chi_\nu(r)\,d\nu,$$

where

$$(3.8) \qquad \chi_\nu(r) = \frac{1}{r}\Big[\cos\theta\; B_\nu^{(1)}(r\sin\theta)A_\nu^{(1)\prime}(r\cos\theta) - \sin\theta\; A_\nu^{(1)}(r\cos\theta)B_\nu^{(1)\prime}(r\sin\theta)\Big]$$

Since for $r \to 0$ $\chi_\nu(r)$ tends to a finite limit, namely, to $-ik(2\nu+1)$, and since the estimate

$$\left|\chi_\nu(r)\right|_{|\tau| \to \infty} = O\left(|\tau| \exp\left\{\sqrt{2k|\tau|}\; r_o\right\}\right),$$

valid in the domain $r \leq r_o$, implies the uniform convergence of the integral (3.7), we may go over to the limit under the integral sign. Thus

$$\lim_{r\to 0} \frac{\partial u}{\partial x} = - \frac{ik\omega I}{2\pi\sqrt{\pi}\, c^2} \int_{-\delta-i\infty}^{-\delta+i\infty} \Gamma(-\tfrac{\nu}{2})\,\Gamma(\tfrac{\nu+1}{2})\,e^{i\pi\nu/2}(2\nu+1)\,\frac{B_\nu^{(3)}(\beta_o)}{B_\nu^{(1)}(\beta_o)}\,d\nu$$

Since this integral is independent of θ, we may conclude that $\partial u/\partial x$ is continuous in the neighborhood of $\alpha = \beta = 0$.

Similar computations show that $\lim\limits_{r\to o} \dfrac{\partial u}{\partial y} = 0$ which proves the continuity of $\partial u/\partial y$ in the neighborhood of the focus. Using similar methods we can prove the continuity of the second derivatives of u with respect to x and y.

We have thus shown that $u(\alpha,\beta)$ is regular in the whole domain $\alpha < A$, $0 \leq \beta \leq \beta_o$.

4. Various representations of the solution

Before we can show that $u(\alpha,\beta)$ is actually the solution to our problem we must represent it in a number of ways.

1. To obtain the first of these representations we complete the path of integration in formula (2.21) by adding to it a circular arc of radius $R \to \infty$ and center $\nu = -1/2$ and utilize the residue theorem. Asymptotic expressions for functions of a parabolic cylinder* show that with

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* cf. (d,3) - (D,5).

$$\alpha < 2\beta_0 - \beta \tag{4.1}$$

the value of the integral over the circular arc tends invariably to zero.

If the inequality (4.1) holds, then, computing the residues at the poles $\nu = 2n$, we find the following representation for u:

$$u = \frac{2\omega i I}{c^2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{B_{2n}^{(3)}(\beta_0)}{B_{2n}^{(1)}(\beta_0)} B_{2n}^{(1)} A_{2n}^{(1)}(\alpha) \tag{4.2}$$

A similar transformation of the contour integral (2.16)-(2.18) for the primary field yields the representation

$$H_0^{(2)}\left(k\,\frac{\alpha^2+\beta^2}{2}\right) = \frac{2i}{\pi^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} B_{2n}^{(3)}(\beta) A_{2n}^{(1)}(\alpha) \tag{4.3}$$

valid under assumption

$$\alpha < \beta\ . \tag{4.4}$$

We observe that there is a simple connection between the functions $A_{2n}^{(1)}(\alpha)$ and Hermite polynomials, namely

$$A_{2n}^{(1)}(\alpha) = (-1)^n \frac{n!}{(2n)!} \exp\left\{-ik\alpha^2/2\right\} H_{2n}\left(\sqrt{ik}\,\alpha\right) \tag{4.5}$$

Thus at least for $\alpha < \beta$ our solution can be represented as a power series in Hermite polynomials $H_{2n}(\sqrt{ik}\,\alpha)$. As for the domain $\alpha < \beta$, direct investigation of the convergence of the series (4.3) shows that for $\alpha > \beta$ this series diverges. Similarly the series (4.2) diverges* for $\alpha > 2\beta_0-\beta$. We may thus claim that the total field can be represented in a series in Hermite polynomials only in the domain defined by the inequality $\alpha < \beta$. In other words, there exists no single analytical representation of the required solution in the form of a series in Hermite polynomials**.

2. In the paper [4] Magnus gave the following solution of the problem under consideration:

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* Convergence of these series was investigated using asymptotic expressions for Hermite polynomials for $n \to \infty$.

** It appears that this fact has not been properly noted in the literature.

$$(4.6)\qquad u = \frac{\omega I}{2\pi c^2}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{B_\nu^{(3)}(\beta_o)}{B_\nu^{(1)}(\beta_o)}B_\nu^{(1)}(\beta)A_\nu^{(3)}(|\alpha|)d\nu \quad (0 < \delta < 1/2)$$

where

$$(4.7)\qquad A_\nu^{(3)}(\alpha) = B_{-\nu-1}^{(3)}(\beta).$$

Since Magnus does not prove that the solution given by him is regular everywhere in the interior of the domain under consideration it is impossible, without further investigation, to conclude that his solution is correct. The need for such investigation is enhanced by the fact that the special solutions $A_\nu^{(3)}(|\alpha|)B_\nu^{(1)}(\beta)$ chosen by Magnus are not regular in the neighborhood of the focus $\alpha = \beta = 0$.

That the solution (4.6) given by Magnus coincides with our solution (2.21) in the domain in which both solutions are regular can be shown by adding a circular arc of radius $R \to \infty$ on the right side of the path of integration in (4.6) (which is permissible in view of the fact that $\alpha < 2\beta_o - \beta$) and by computing the residues at the poles $\nu = 2n$. These computations lead to the formula (4.2). In view of the well known properties of the solutions of elliptic equations we may thus conclude that the solutions (4.6) and (4.2) coincide in their common domain of regularity.

Finally we observe that the field of the source can also be represented as an integral in the functions $A_\nu^{(3)}(|\alpha|)$:

$$(4.8)\qquad H_o^{(2)}\left(k\,\frac{\alpha^2+\beta^2}{2}\right) = \frac{1}{2\pi^2}\int_{\delta-i\infty}^{-\delta+i\infty}B_\nu^{(3)}(\beta)A_\nu^{(3)}(|\alpha|)d\nu \quad (0 < \delta < 1),$$

and this representation is valid in the whole domain with the exception of the line* $\beta = 0$

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* cf. [4]

3. On the basis of the results in section 2 the complete solution E can be represented by means of the integral*

$$(4.9)\qquad E = \frac{\omega I}{2\pi c^2}\int_{-\delta-i\infty}^{-\delta+i\infty}\left[B_\nu^{(1)}(\beta_o)B_\nu^{(2)}(\beta)-B_\nu^{(2)}(\beta_o)B_\nu^{(1)}(\beta)\right]\frac{A_\nu^{(3)}(|\alpha|)}{B_\nu^{(1)}(\beta_o)}\,d\nu \quad (\beta > 0),$$

which permits us to obtain another form of the solution of great use in the sequel.

We now complete the path of integration by addition on the left of a circular arc of radius $R \to \infty$ and we use the residue theorem. Since the singularities of the integrand are the roots of the equation

$$(4.10)\qquad B_{\nu_n}^{(1)}(\beta_o) = 0 \qquad (n = 1, 2, 3, \ldots),$$

located (see supplement 2) on the line $\mathrm{Re}\left\{\nu\right\} = -1/2$, and since $\mathrm{Im}\left\{\nu_1\right\} < \mathrm{Im}\left\{\nu_2\right\} < \ldots$, and, finally, since the integral along the arc tends to zero, it follows that

$$(4.11)\qquad E = \frac{2\sqrt{\pi}\, ikI}{c}\sum_{n=1}^{\infty}\frac{B_{\nu_n}^{(2)}(\beta_o)B_{\nu_n}^{(1)}(\beta)}{\left[dB_\nu^{(1)}(\beta_o)/d\nu\right]_{\nu=\nu_n}}\,A_{\nu_n}^{(3)}(|\alpha|).$$

Computation of the derivative which appears in the denominator leads to the following representation of E in a series in the functions $B_{\nu_n}^{(1)}(\beta)$:

$$(4.12)\qquad E(\alpha,\beta) = -\frac{I\sqrt{\pi}\, ik}{c}\sum_{n=1}^{\infty} B_{\nu_n}^{(1)}(\beta)A_{\nu_n}^{(3)}(|\alpha|)\Big/\int_0^{\beta_o} B_{\nu_n}^{(1)2}(\beta)\,d\beta\ .$$

This form of the solution will be utilized in the next section when we prove that the solution of our problem satisfies the boundary condition and the radiation principle.

We wish to note that the representation (4.12) can be obtained directly by integrating the wave equation

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* Here we made use of the relation:

$$B_\nu^{(3)}(\beta_o)B_\nu^{(1)}(\beta) - B_\nu^{(1)}(\beta_o)B_\nu^{(3)}(\beta) = 2\sqrt{\pi}\left[B_\nu^{(1)}(\beta_o)B_\nu^{(2)}(\beta)-B_\nu^{(2)}(\beta_o)B_\nu^{(1)}(\beta)\right]$$

$$(4.13) \qquad (\partial^2 E/\partial\alpha^2) + (\partial^2 E/\partial\beta^2) + k^2(\alpha^2+\beta^2)E = 4\pi i\omega c^{-2}(\alpha^2+\beta^2)j \ ,$$

under the assumption that the intensity of the current is uniformly distributed over a small neighborhood of the focus:

$$j(\alpha,\beta) = \begin{cases} I/2\ (\alpha^2+\beta^2)\alpha_1\beta_1 & \text{for } |\alpha| < \alpha_1,\ 0 \leq \beta \leq \beta_1 \\ 0 \text{ for the remaining values of } \alpha \text{ and } \beta \end{cases}$$

with α_1 and β_1 tending to zero.

5. Test of the formal solution

The results of section 3 imply that the function u constructed above and defined by formula (2.21) is a regular solution of Helmholtz's equation at every point of the domain under consideration. As regards the complete solution E it is the sum of the field of a line source through the focus and the function u and as such it obviously satisfies the Helmholtz equation at all points of the domain other than the focus where it has a singularity of the required type.

It therefore remains to check if the solution $E(\alpha,\beta)$ satisfies the boundary condition and the radiation condition. To do this it is best to use the form (4.12) of the solution.

1. The eigenfunctions $B^{(1)}_{\nu_n}(\beta)$ satisfy the condition $B^{(1)}_{\nu_n}(\beta_0) = 0$. Consequently in order to check if the boundary condition $E(\alpha,\beta_0) = 0$ is satisfied it suffices to show that the series (4.12) converges uniformly on the interval $0 \leq \beta \leq \beta_0$.

Using the asymptotic expressions for the functions $B^{(1)}_{\nu}(\beta)$ and $A^{(3)}_{\nu}(\alpha)$ for $\nu_n = -1/2 + i\tau_n\ (\tau_n \to \infty)$:

$$(5.1) \qquad B^{(1)}_{\nu_n}(\beta) \approx \cos\sqrt{2ki\tau_n}\,\beta; \quad A^{(3)}_{\nu_n}(\alpha) \approx 2\sqrt{2\pi i/\tau_n}\,\exp\left\{-\sqrt{2k\tau_n}\,\alpha\right\},$$

and the formula ($D_2 2$)

$$(5.2)\qquad \tau_n\Big|_{n\to\infty} \approx \frac{1}{2k\beta_o^2}\left(\frac{2n-1}{2}\pi\right)^2,$$

we find that the terms of the series (4.12) are of the order of

$$n^{-1}\exp\left\{-\frac{2n-1}{2\beta_o}\pi\alpha\right\},$$

which implies the uniform convergence of the series in the indicated domain. This shows that our solution satisfies the prescribed boundary condition.

2. To see that our solution satisfies the radiation condition (1.9) it suffices to show that the following two equalities hold

$$(5.3)\qquad |u|_{\alpha\to\infty} = O(\alpha^{-1/2});\qquad \left|\frac{1}{\alpha}\frac{\partial u}{\partial \alpha} + iku\right|_{\alpha\to\infty} = O(\alpha^{-1/2}).$$

From the estimates* ($n\to\infty$)

$$(5.4)\qquad \left|\frac{B^{(1)}_{\nu_n}(\beta)}{\int_0^{\beta_o} B^{(1)2}_{\nu_n}(\beta)d\beta}\right| = O(1),\qquad |A^{(3)}_{\nu_n}(\alpha)| = \alpha^{-1/2}O(n^{-5/2}),$$

we conclude that the first part of the radiation condition is satisfied.

Differentiating the series (4.12) and using the estimate (derived in supplement 3)

$$(5.5)\qquad \left|\frac{1}{\alpha}A^{(3)\prime}_{\nu_n}(\alpha) + ikA^{(3)}_{\nu_n}(\alpha)\right|_{n\to\infty} = \alpha^{-5/2}O(n^{-5/2})$$

which is valid for arbitrary α we find that

---

[*] The second estimate (5.4), derived in supplement 3, is uniform with respect to α

$$\left| \frac{1}{\alpha} \frac{\partial u}{\partial \alpha} + iku \right|_{\alpha \to \infty} = O(\alpha^{-5/2})$$

which proves that the second part of the radiation condition is satisfied.

Thus the formal solution of our problem obtained in section 2 satisfies all conditions specified in section 1 where the problem was mathematically formulated.

6. The uniqueness theorem

A solution of the Helmholtz equation

$$\Delta v + k^2 v = 0, \tag{6.1}$$

which is regular in the domain (D) bounded by a branch of the parabola $\beta = \beta_o$ and the axis of abscissas and satisfies the boundary conditions

$$v\big|_{\beta=\beta_o} = 0, \qquad \partial v/\partial y\big|_{y=0} = 0 \tag{6.2}$$

and the radiation conditions*

$$v\big|_{\alpha \to \infty} = O(\alpha^{-1/2}); \qquad \frac{1}{\alpha}\frac{\partial v}{\partial \alpha} + ikv\Big|_{\alpha \to \infty} = 0(\alpha^{-1/2}), \tag{6.3}$$

vanishes identically.

For proof we consider the domain (D') bounded by the x-axis, by a branch of the parabola $\beta = \beta_o$ and by a branch of the parabola $\alpha = A$ orthogonal to the parabola $\beta = \beta_o$. Multiplying equation (6.1) by the conjugate function $\bar{v}$ and the conjugate equation by v and integrating the difference of the two equations over (D') we find, using Green's theorem and the conditions (6.2), that

$$\int_0^{\beta_o} \left(\bar{v} \frac{\partial v}{\partial \alpha} - v \frac{\partial \bar{v}}{\partial \alpha} \right)_{\alpha=A} d\beta = 0. \tag{6.4}$$

Using a consequence of the radiation condition

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* Uniformly with respect to the variable β.

$$(6.5)\qquad \frac{1}{\alpha}\left(\bar{v}\,\frac{\partial v}{\partial \alpha} - v\,\frac{\partial \bar{v}}{\partial \alpha}\right) = -\,2ik|v|^2 + \bar{v}o(\alpha^{-1/2}) - vo(\alpha^{-1/2})$$

and noting that for $\alpha \rightarrow \infty$ the products $\alpha^{1/2}v$ and $\alpha^{1/2}\bar{v}$ remain bounded we obtain from (6.4) the following equality:

$$(6.6)\qquad \lim_{\alpha\rightarrow\infty} \int_0^{\beta_o} \alpha|v|^2\, d\beta = 0.$$

To show that (6.6) implies that $v \equiv 0$ we consider the following special solutions of the Helmholtz equation:

$$(6.7)\qquad u_n = A^{(3)}_{\nu_n}(\alpha)B^{(1)}_{\nu_n}(\beta),$$

where the ν_n are roots of the equation $B^{(1)}_{\nu_n}(\beta_o) = 0$. These solutions are regular in the domain (D'') bounded by the parabolas $\beta = \beta_o$, $\alpha = \varepsilon > 0$, $\alpha = A$, and the x-axis, and satisfy the boundary conditions

$$(6.8)\qquad u_n\big|_{\beta=\beta_o} = 0; \qquad \partial u_n/\partial\beta\big|_{\beta=0} = 0$$

and the radiation condition (6.3).

Applying Green's formula to the pair of functions v, u_n we obtain the relation

$$(6.9)\qquad \int_0^{\beta_o}\left(u_n\,\frac{\partial v}{\partial\alpha} - v\,\frac{\partial u_n}{\partial\alpha}\right)_{\alpha=\varepsilon} d\beta = \int_0^{\beta_o}\left(u_n\,\frac{\partial v}{\partial\alpha} - v\,\frac{\partial u_n}{\partial\alpha}\right)_{\alpha=A} d\beta.$$

This relation implies that the value of the integral

$$\int_0^{\beta_o}\left(u_n\,\frac{\partial v}{\partial\alpha} - v\,\frac{\partial u_n}{\partial\alpha}\right) d\beta$$

is independent of α. In view of the fact that this integral tends to zero as $\alpha \rightarrow \infty$ (a consequence of the radiation condition) we conclude that

$$\int_0^{\beta_o} \left(u_n \frac{\partial v}{\partial \alpha} - v \frac{\partial u_n}{\partial \alpha} \right) d\beta = 0 \qquad \text{for all } \alpha. \tag{6.10}$$

Substituting for u_n its value and putting

$$v_n = \int_0^{\beta_o} v\, B_{\nu_n}^{(1)} d\beta \Big/ \int_0^{\beta_o} B_{\nu_n}^{(1)^2}\, d\beta \qquad (n = 1, 2, 3, \ldots). \tag{6.11}$$

whence

$$A_{\nu_n}^{(3)}(\alpha) \frac{dv_n}{d\alpha} = v_n \frac{dA_{\nu_n}^{(3)}(\alpha)}{d\alpha} \tag{6.12}$$

$$v_n = c_n A_{\nu_n}^{(3)}(\alpha).$$

where c_n is a constant.

Since the system of functions $B_{\nu_n}^{(1)}(\beta)$ is closed[*] , i.e.,

$$\sum_{n=1}^{\infty} |v_n|^2 \int_0^{\beta_o} B_{\nu_n}^{(1)^2} d\beta = \int_0^{\beta_o} |v|^2\, d\beta, \tag{6.13}$$

we conclude from (6.6) that $\lim\limits_{\alpha \to \infty} \alpha |v_n|^2 = 0$. In view of (6.12) this means that

$$|c_n|^2 \lim_{\alpha \to \infty} \alpha |A_{\nu_n}^{(3)}|^2 = 0. \tag{6.14}$$

Now the asymptotic representation of the functions $A_{\nu_n}^{(3)}(\alpha)$ for $\alpha \to \infty$[**] shows that the product $\alpha |A_{\nu_n}^{(3)}(\alpha)|^2$ does not tend to zero for $\alpha \to \infty$. Hence $c_n = 0$, i.e.,

$$v_n = 0; \qquad n = 1, 2, 3, \ldots \ . \tag{6.15}$$

In view of the closedness condition (6.13) $v \equiv 0$ for all $\alpha > 0$ and $0 \leq \beta \leq \beta_o$, which is what we wished to prove.

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[*] This follows from the general theory of eigenfunctions associated with second order differential equations

[**] cf. ($D_1$2)

7. Investigation of the solution for large values of the wave number k. Transition to geometrical optics.

We now investigate the behavior of the solution obtained in section 2 as $k \to \infty$ which should correspond to the case of geometrical optics. To study the behavior of the secondary field as $k \to \infty$ we make use of the following asymptotic formulas

$$(7.1)\qquad B_\nu^{(1)}(\beta)\Big|_{k\to\infty} \approx \frac{\sqrt{\pi}}{\Gamma((\nu+1)/2)}\,(ik\beta^2)^{\nu/2}e^{ik\beta^2/2};$$

$$B_\nu^{(3)}(\beta)\Big|_{k\to\infty} \approx \Gamma\left(\frac{\nu+1}{2}\right)(ik\beta^2)^{-(1+\nu)/2}e^{-ik\beta^2/2},\quad R\ \nu > -\frac{1}{2},$$

which are obtained from $(D_1 1)$ and $(D_1 2)$ by replacing ν by $-\nu-1$ and by noting that $A_\nu(\alpha) = B_{-\nu-1}(\alpha)$.

Substituting (7.1) into (2.21) yields the relation

$$(7.2)\qquad u(\alpha,\beta)\Big|_{k\to\infty} \approx \frac{I\sqrt{k}\,e^{-i\pi/4}}{2\pi\sqrt{\pi}\,c\beta_o}\exp\left\{\frac{ik}{2}(\beta^2-2\beta_o^2)\right\} I_1;$$

$$I_1 = \int_{-\delta-i\infty}^{-\delta+i\infty}\Gamma\left(-\frac{\nu}{2}\right)\Gamma^2\left(\frac{\nu+1}{2}\right)k^{-\nu/2}\left(\frac{\beta_o^2}{\beta}\right)^{-\nu}e^{i\pi\nu/4}A_\nu^{(1)}(\alpha)\,d\nu.$$

To compute the integral I_1 we make use of the integral representation $(D_1 6)$, which when put in (7.2) yields:

$$(7.3)\qquad I_1 = \frac{\sqrt{2k}}{c\beta_o}\exp\left\{-\frac{i\pi}{4}-\frac{ik}{2}(\beta^2-2\beta_o^2)\right\}\int_{-\infty}^{\infty}\exp\left\{\frac{\theta}{2}-\frac{ik\alpha^2}{2}\,\mathrm{th}\,\theta\right\}\frac{I_2\,d\theta}{\sqrt{\mathrm{ch}\,\theta}};$$

$$I_2 = \frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\Gamma\left(\frac{\nu+1}{2}\right)e^{\nu\theta}\,k^{-\nu/2}\left(\frac{\beta_o^2}{\beta}\right)^{-\nu}e^{i\pi\nu/4}\,d\nu.$$

We now make the substitution $\nu = 2\mu-1$ in the complex integral. If we put $2\delta' = 1-\delta$, we find that

$$I_2 = \frac{2\beta_o^2}{\sqrt{k\beta}} \exp\left(-\frac{i\pi}{4} - \theta\right) \frac{1}{2\pi i} \int_{\delta'-i\infty}^{\delta'+i\infty} \Gamma(\mu)\left(-ike^{-2\theta}\frac{\beta_o^4}{\beta^2}\right)^{-\mu} d\mu,$$

whence*

$$(7.4) \qquad I_2 = 2\sqrt{k}\ \left(\frac{\beta_o^2}{\beta}\right) \exp\left\{-\frac{i\pi}{4} - \theta + ik\frac{\beta_o^4}{\beta^2}e^{-2\theta}\right\} .$$

Substituting (7.4) into (7.3) and effecting the change of variable $e^{-\theta} = x$ in the integral I_1 we find that

$$(7.5) \qquad u\Big|_{k\to\infty} \approx \frac{4kI\beta_o}{c\beta} \exp\left\{\frac{ik}{2}(\beta^2-2\beta_o^2)\right\} \int_0^\infty \exp\left\{-\frac{ik\alpha^2}{2}\frac{1-x^2}{1+x^2} + ik\frac{\beta_o^4}{\beta^2}x^2\right\} \frac{dx}{\sqrt{1+x^2}} .$$

As $k \to \infty$ the principal contribution in the latter integral comes from the neighborhood of the point $x = 0$. Hence

$$(7.6) \qquad u\Big|_{k\to\infty} = \frac{1}{\sqrt{c}}\, 2I(\pi k)^{1/2}\left[\beta_o^2 + (\alpha\beta/\beta_o)^2\right]^{-1/2} \exp\left\{\frac{i\pi}{4} - ik\beta_o^2 - \frac{ik}{2}(\alpha^2-\beta^2)\right\} .$$

If we now revert to Cartesian coordinates and note the relation (cf. Fig. 2)

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* Here we have utilized the relation

$$e^{-x} = \frac{1}{2\pi i}\int_{\delta'-i\infty}^{\delta'+i\infty} \Gamma(\mu)\, x^{-\mu}\, d\mu .$$

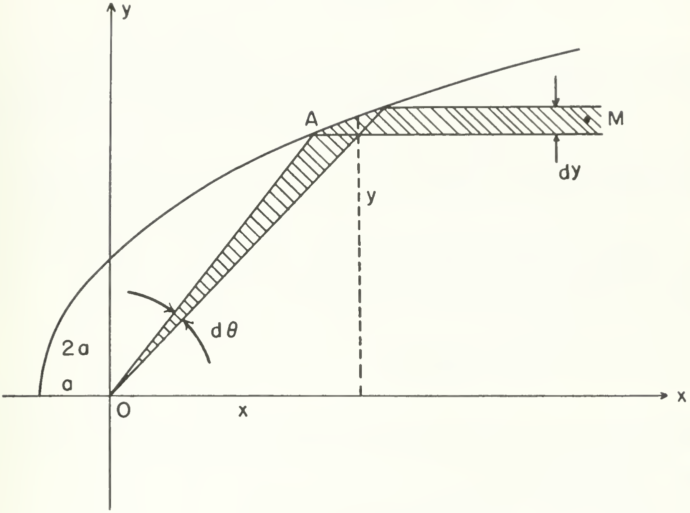

Figure 2

we can rewrite formula (7.6) as follows:

$$(7.7) \qquad u|_{k\to\infty} \approx \frac{2I}{c\beta_o}\left(\frac{\pi k}{1+(y/\beta_o)^2}\right)^{1/2} e^{i\pi/4-iks} .$$

From the latter relation it is apparent that the exponent s which appears in the phase factor represents the path OAM which would be traversed by a ray emanating from the origin O and passing through M as a result of reflection from the surface of the parabolic cylinder, such reflection obeying the laws of geometrical optics.

It is possible to show that the amplitude of the factor in formula (7.7) also agrees with the approximation yielded by geometrical optics. Indeed, if we compute the mean density p over a period of the energy flow at some point M*:

$$(7.8) \qquad p = (c/16\pi)(u^*H_y - uH_y^*),$$

and if we make use of the asymptotic representation for H_y (which is identical with (7.7)), we find that

$$(7.9) \qquad p|_{k\to\infty} \approx (kI^2/2c\beta_o^2) \Big/ \left[I + (y/\beta_o^2)^2\right],$$

On the other hand, we are led to this very expression by assuming the laws of geometrical optics. To see this we compute the electromagnetic energy $\frac{W}{2\pi}\,d\theta = \frac{W}{2\pi|dy/d\theta|}\,|dy| = p|dy|$ which passes in a unit of time through a small area of height dy in the vicinity of the point M (cf. Fig. 2). Here $W = \sqrt{I}\,kI^2/2c$ denotes the average, with respect to time, amount of energy radiated by a unit length of the source. Using the geometrical relations

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* Here we discard the component H_x of the secondary magnetic field since the latter is of the order of $1/k$ in comparison with H_y.

$\mathrm{tg}\theta = y/x, \quad y^2 = 2\beta_o^2(x+\beta_o^2/2)$ we find that

$$y = \beta_o^2 \,\mathrm{ctg}\, \frac{\theta}{2} \,; \quad \frac{dy}{d\theta} = -\frac{\beta_o^2}{2}\left[1 + \left(\frac{y}{\beta_o}\right)^2\right],$$

which leads to the expression (7.9) for p.

Thus in the limiting case $k \to \infty$ the solution of our problem goes over into the approximation yielded by geometrical optics.

8. Generalization of the solution to the case of arbitrary location of the source on the axis of the cylinder.

In the case when the source of vibrations is not located at the focus the method of solution presented in section 2 leads to complex computations connected with the representation of the field of the source as an integral in the functions of a parabolic cylinder. In such cases it is expedient to look at once for the complete field.

We first consider the case when the source is located at an arbitrary point $\alpha = \alpha^*$, $\beta = 0$ of the positive part of the axis of abscissas. Replacing the linear current I by a current equal to it but distributed uniformly over a small area we can write the following non-homogeneous equation.

$$(8.1) \qquad \frac{\partial^2 E}{\partial \alpha^2} + \frac{\partial^2 E}{\partial \beta^2} + k^2(\alpha^2 + \beta^2)E = \frac{4\pi i\omega}{c^2}(\alpha^2 + \beta^2)j(\alpha,\beta)$$

$$(0 < \alpha < \infty, \quad 0 < \beta < \beta_o);$$

$$j(\alpha,\beta) = \begin{cases} I/\left[4(\alpha^2+\beta^2)\varepsilon\gamma\right] & \text{for} \quad \alpha^* - \varepsilon < \alpha < \alpha^* + \varepsilon, \quad 0 < \beta < \gamma \\ 0 \quad \text{otherwise} \end{cases}$$

and the boundary conditions

$$(8.2) \qquad E\Big|_{\beta=\beta_o} = 0, \quad \partial E/\partial\alpha\Big|_{\alpha=0} = 0, \quad \partial E/\partial\beta\Big|_{\beta=0} = 0 .$$

If we seek the function $E(\alpha,\beta)$ in the form

$$(8.3) \quad E = \frac{\sqrt{k}\, e^{-i\pi/4}}{2\pi^2} \int_{-\delta-i\infty}^{\delta+i\infty} E_\nu(\beta)\,\Gamma\!\left(-\frac{\nu}{2}\right)\Gamma\!\left(\frac{\nu+1}{2}\right)e^{i\pi\nu/2} A_\nu^{(1)}(\alpha)\,d\nu, \quad 0 < \delta < 1/2 ,$$

where

$$E_\nu(\beta) = \int_0^\infty EA_\nu^{(1)}(\alpha)d\alpha, \tag{8.4}$$

then, multiplying (8.1) by $A_\nu^{(1)}(\alpha)$ and integrating* with respect to α from 0 to ∞ we obtain for $E_\nu(\beta)$ an ordinary differential equation. Let $E_\nu(\beta_o)=0$, $E_\nu'(0) = 0$. Under the assumption that ϵ, $\gamma \rightarrow 0$, these conditions imply that

$$E_\nu(\beta) = \frac{I\sqrt{\pi k}}{c} B_\nu^{(1)}(\beta)B_\nu^{(3)}(\beta_o) - B_\nu^{(3)}(\beta)B_\nu^{(1)}(\beta_o) \frac{A_\nu^{(1)}(\alpha^*)}{B_\nu^{(1)}(\beta_o)} . \tag{8.5}$$

Substituting (8.5) in (8.3) we obtain the solution of our problem:

$$\begin{aligned} E(\alpha,\beta) = \frac{Ik}{2\pi\sqrt{\pi}\, c} \int_{-\delta-i\infty}^{-\delta+i\infty} & \Gamma(-\tfrac{\nu}{2})\Gamma(\tfrac{\nu+1}{2})e^{i\pi\nu/2}\Big[B_\nu^{(1)}(\beta)B_\nu^{(3)}(\beta_o) - \\ & - B_\nu^{(3)}(\beta)B_\nu^{(1)}(\beta_o)\Big] \frac{A_\nu^{(1)}(\alpha^*)}{B_\nu^{(1)}(\beta_o)} A_\nu^{(1)}(\alpha)d\nu. \end{aligned} \tag{8.6}$$

An analogous procedure can be used in the case when the source is located to the left of the focus at an arbitrary point of the negative part of the axis of abscissas.

In the preceeding discussion we have restricted ourselves to the case when the source is located on the axis of the cylinder. A similar discussion could be applied in the case when the source is located in an arbitrary position inside the cylinder. In this more general case one would have to introduce in addition to the even functions $A_\nu^{(1)}(\alpha)$ also odd functions for which it would be necessary to derive the corresponding representations.

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* Here we make use of the radiation condition

Supplements

1. Some properties of the functions of the parabolic cylinder.

Asymptotic behavior for large values of the argument:

$$(D_1 1)\quad A_\nu^{(1)}(\alpha)\Big|_{|\alpha|\to\infty} = e^{ik\alpha^2/2}\,\frac{\Gamma(1/2)}{\Gamma(-\frac{\nu}{2})}(ik\alpha^2)^{-(\nu+1)/2}\left[1+O\left(\frac{1}{k\alpha^2}\right)\right] + \exp\left\{-\frac{ik\alpha^2}{2}-\frac{i\pi\nu}{2}\right\}\frac{\Gamma(1/2)}{\Gamma[(\nu+1)/2]}(ik\alpha^2)^{\nu/2}\left[1+O\left(\frac{1}{k\alpha^2}\right)\right];$$

$$(D_1 2)\quad A_\nu^{(3)}(\alpha)\Big|_{|\alpha|\to\infty} = (ik\alpha)^{\nu/2}\,\Gamma\left(-\frac{\nu}{2}\right)e^{-ik\alpha^2/2}\left[1+O\left(\frac{1}{k\alpha^2}\right)\right]$$

Asymptotic representations as $|\nu|\to\infty$:

$$(D_1 3)\quad A_\nu^{(1)}(\alpha) = \cos\sqrt{ik(2\nu+1)}\;\alpha\left[1+O\left(\frac{1}{\sqrt{2\nu+1}}\right)\right];$$

$$(D_1 4)\quad B_\nu^{(1)}(\beta) = \operatorname{ch}\sqrt{ik(2\nu+1)}\;\beta\left[1+O\left(\frac{1}{\sqrt{2\nu+1}}\right)\right];$$

$$(D_1 5)^*\quad B_\nu^{(3)}(\beta) = \mp\frac{2\sqrt{\pi}}{\sqrt{2\nu+1}}\exp\left\{\pm\sqrt{ik(2\nu+1)}\;\beta\right\}.$$

Integral representations

$$(D_1 6)^{**}\quad A_\nu^{(1)}(\alpha) = \frac{\sqrt{2\pi}}{\Gamma(-\frac{\nu}{2})\Gamma(\frac{\nu+1}{2})}\int_{-\infty}^{\infty}\exp\left\{\left(\nu+\frac{1}{2}\right)\theta-\frac{ik\alpha^2}{2}\operatorname{th}\theta\right\}\frac{d\theta}{\sqrt{\operatorname{ch}\theta}}$$

$$(-1<\operatorname{Re}\nu<0);$$

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[*] Here the upper sign refers to the domain $\operatorname{Re}\{\nu\} < -\frac{1}{2}$ and the lower sign to the domain $\operatorname{Re}\{\nu\} > -\frac{1}{2}$.

[**] This formula is obtained from the well-known integral representation [10]

$$F(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{tz}dt;\ (\operatorname{Re}b > \operatorname{Re}a > 0)$$

by the change of variable $(1-t)/t = e^{2\theta}$

$$(D_1 7) \qquad A_\nu^{(3)}(\alpha) = \frac{2^{\nu+1}\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})}(k\alpha^2)^{\nu/2}\exp\left\{\frac{i\pi\nu}{4} - \frac{ik\alpha^2}{2}\right\}\int_0^\infty \exp\left\{\frac{ix^2}{4k\alpha^2} - x\right\} x^{-\nu-1}dx$$

$$(\mathrm{Re}\,\nu < 0),$$

2. Distribution of the zeros of the function $B_\nu^{(1)}(\beta)$

To investigate the locations of the roots of the equation

$$(D_2 1) \qquad B_\nu^{(1)}(\beta) = 0 \quad (\nu = \sigma + i\tau),$$

we make use of the relation

$$\overline{B}_\nu^{(1)}(\beta)B_\nu^{(1)\prime}(\beta) - \overline{B}_\nu^{(1)\prime}(\beta)B_\nu^{(1)}(\beta) = 2ik(1+2\sigma)\int_0^\beta |B_\nu^{(1)}|^2\,d\beta,$$

which can be gotten from the differential equations and the initial conditions for the functions under consideration. If we assume that for a given β the value of ν is such that $B_\nu^{(1)}(\beta) = 0$, then $\overline{B}_\nu^{(1)}(\beta) = 0$. Consequently, $\mathrm{Re}\{\nu\} = \sigma = -1/2$, i.e., all roots ν_n of the function $B_\nu^{(1)}(\beta)$ are located on the line $\mathrm{Re}\{\nu\} = -1/2$.

A more detailed investigation shows that the imaginary parts of these roots $|\tau_n| \to \infty$, are given asymptotically by

$$(D_2 2) \qquad \tau_n \approx \frac{1}{2k\beta^2}\left(\frac{2n-1}{2}\pi\right)^2 .$$

It can also be shown that the numbers τ_n are bounded from below.

3. Derivation of a certain estimate for the function $A_\nu^3(\alpha)$

It is the purpose of this supplement to establish a certain estimate for the function of the third kind $A_\nu^{(3)}(\alpha)$ valid for $-T < \mathrm{Im}\{\nu\} < \infty$ (T is a large positive number) uniformly with respect to α.

Integrating $(D_1 7)$ by parts we find that

$$A_\nu^{(3)}(\alpha) = \frac{2^{\nu+1}\sqrt{\pi}}{\Gamma(1-\nu/2)} (k\alpha^2)^{\nu/2} \exp\left\{ \frac{i\pi\nu}{4} - \frac{ik\alpha^2}{2} \right\} \times$$

$$\times \frac{1}{\nu} \int_0^\infty x^{-\nu} \exp\left\{ \frac{ix^2}{4\alpha^2 k} - x \right\} \left(\frac{ix}{2k\alpha^2} - 1\right) dx; \quad \nu = -\delta + i\tau \; ; \; \delta > 0.$$

Estimating the absolute value of the latter integral and taking into consideration the behavior of the Γ-function for large τ we find that

$$(D_3 1) \qquad |A_\nu^{(3)}(\alpha)| = \alpha^{-\delta}\, \phi(\tau) \qquad (\delta > 0),$$

and

$$(D_3 2) \qquad \phi(\tau) = \begin{cases} O(1) & \text{for} \quad -T \leq \tau \leq T, \\ O(\tau^{-1-\delta/2}) & \text{for} \quad T < \tau < \infty. \end{cases}$$

Further, using an integral representation for the linear combination $\alpha^{-1} A_\nu^{(3)\prime}(\alpha) + ikA_\nu^{(3)}(\alpha)$ (such a representation can be easily obtained from ($D_1 7$)) we find, proceeding as above, that

$$(D_3 3) \qquad \left| \frac{1}{\alpha} A_\nu^{(3)}(\alpha) + ikA_\nu^{(3)}(\alpha) \right| = \alpha^{-2-\delta}\, \Psi(\tau),$$

where

$$(D_3 4) \qquad \Psi(\tau) = \begin{cases} O(1) & \text{for} \quad -T \leq \tau \leq T, \\ O(\tau^{-1-\delta/2}) & \text{for} \quad T < \tau < \infty. \end{cases}$$

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THE ELECTROMAGNETIC FIELD OF A DIPOLE LOCATED IN THE INTERIOR OF A PARABOLIC REFLECTOR

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with Appendix by W. Magnus and N.N. Lebedev)

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This paper is devoted to the problem of finding the electromagnetic field of a dipole located inside a parabolic reflector when the radiating dipole is perpendicular to the axis of the paraboloid. This problem was considered earlier by Pinney[2]. However, from the mathematical standpoint, Pinney's representation of the solution in the form of series of Laguerre polynomials is not sufficiently well justified (cf. Section 3 below).

The final solution of the problem, applicable everywhere in the interior of the paraboloid, is presented in this paper by means of formulas which express the required field components in the form of complex integrals. The exact solution is used, among others, to find the field in the limiting case of geometrical optics when the wavelength is much shorter than the focal distance of the paraboloid.

1. Formulation of the problem and representation of the solution by means of complex integrals

We wish to find the electromagnetic field due to a dipole of moment p located at the focus of an ideally conducting infinite paraboloid of rotation and perpendicular to its axis. We select a Cartesian coordinate system whose z-axis coincides with the axis of the paraboloid, whose x-axis coincides with the direction of the dipole, and whose origin coincides with the focus O (Fig. 1).

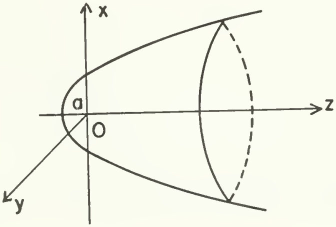

Figure 1

It follows from Maxwell's equations that each of the six components of the electromagnetic field is a solution of Helmholtz's equation. If we put[*]

$$E_x = u + w \cos 2\phi,$$

$$E_y = w \sin 2\phi, \tag{1.1}$$

$$E_z = v \cos \phi,$$

where $u = u(\alpha,\beta)$, $v = v(\alpha,\beta)$, $w = w(\alpha,\beta)$, then, using paraboloid coordinates (α,β,ϕ), we are led to the following expressions for the unknown functions[**] u, v, and w :

$$\frac{1}{\alpha}\frac{\partial}{\partial\alpha}\alpha\frac{\partial u}{\partial\alpha} + \frac{1}{\beta}\frac{\partial}{\partial\beta}\beta\frac{\partial u}{\partial\beta} + k^2(\alpha^2+\beta^2)u = 0,$$

$$\frac{1}{\alpha}\frac{\partial}{\partial\alpha}\alpha\frac{\partial v}{\partial\alpha} + \frac{1}{\beta}\frac{\partial}{\partial\beta}\beta\frac{\partial v}{\partial\beta} + k^2(\alpha^2+\beta^2)v - \left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)v = 0, \tag{1.2}$$

$$\frac{1}{\alpha}\frac{\partial}{\partial\alpha}\alpha\frac{\partial w}{\partial\alpha} + \frac{1}{\beta}\frac{\partial}{\partial\beta}\beta\frac{\partial w}{\partial\beta} + k^2(\alpha^2+\beta^2)w - 4\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)w = 0,$$

where $k = \frac{\omega}{c}$ = the wave number.

In view of the fact that on the surface of an ideal conductor the tangential components of the electromagnetic field vanish, we have, in the case

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[*] The nature of the dependence of the required quantities on the angle ϕ can be predicted from the expressions for the components of the field of the dipole (cf. [1,5]).

[**] Note that in a paraboloid system of coordinates (α,β,ϕ) defined by the relations: $x = \alpha\beta\cos\phi$, $y = \alpha\beta\sin\phi$, $z = \frac{1}{2}(\alpha^2 - \beta^2)$, where $0 \le \alpha < \infty$, $0 \le \beta < \infty$, the Helmholtz equation for the i-th component of the field takes the form

$$\frac{1}{\alpha}\frac{\partial}{\partial\alpha}\alpha\frac{\partial E_i}{\partial\alpha} + \frac{1}{\beta}\frac{\partial}{\partial\beta}\beta\frac{\partial E_i}{\partial\beta} + \left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\frac{\partial^2 E_i}{\partial\phi^2} + k^2(\alpha^2+\beta^2)E_i = 0.$$

under consideration, the following two boundary conditions:

$$(1.3) \qquad E_\phi\big|_{\beta=\beta_0} = 0, \qquad E_\alpha\big|_{\beta=\beta_0} = 0,$$

where $\beta = \beta_0$ is the equation of the paraboloid and the interior of the paraboloid is described by $0 \leq \alpha < \infty$, $0 \leq \beta \leq \beta_0$.

The well-known relations which connect the components of a vector in a paraboloid system of coordinates with its components in a Cartesian coordinate system allow us to obtain from (1.3) boundary conditions for the required functions u,v, and w. Indeed, since

$$E_\phi = E_y \cos\phi - E_x \sin\phi, \quad \text{whereas} \quad E_\alpha = \frac{1}{\sqrt{\alpha^2+\beta^2}}\left[\alpha\, E_z + \beta(E_x \cos\phi + E_y \sin\phi)\right],$$

it follows that on the surface of the paraboloid $\beta = \beta_0$ the following linear relations connect u, v, and w:

$$(1.4) \qquad w|_{\beta=\beta_0} = u|_{\beta=\beta_0}, \quad \alpha v|_{\beta=\beta_0} + 2\beta_0 u|_{\beta=\beta_0} = 0.$$

We represent the required field as the sum of the field of the source in free space and the secondary field, which we shall seek in the form of an integral of the appropriate sets of special solutions of (1.2). Thus the complete field will be written in the form

$$E_x = E_x^0 + E_x^1, \quad E_y = E_y^0 + E_y^1, \quad E_z = E_z^0 + E_z^1,$$

where, in the notations of (1.1), the primary field (i.e., the dipole field) is given by means of the formulas:

$$u^{o} = \frac{p}{ikc}\,\frac{e^{-ikr}}{r^{3}}\left[\frac{\rho^{2}}{2r^{2}}\,(3 + 3ikr - k^{2}r^{2}) - (1 + ikr - k^{2}r^{2})\right],$$

$$v^{o} = \frac{p}{ikc}\,\frac{e^{-ikr}}{r^{3}}\,\frac{\rho z}{r^{2}}\,(3 + 3ikr - k^{2}r^{2}),$$

(1.5)

$$w^{o} = \frac{p}{ikc}\,\frac{e^{-ikr}}{r^{3}}\,\frac{\rho^{2}}{2r^{2}}\,(3 + 3ikr - k^{2}r^{2}),$$

$$\left(r = \frac{1}{2}\,(\alpha^{2} + \beta^{2}),\ \rho = \alpha\beta \right).$$

In the case under discussion, a convenient set of special solutions of (1.2) regular in the interior of the paraboloid is the set of solutions of the form

$$u_{\nu} = a_{o}(\nu)B^{(1)}_{\nu,0}\,(\beta)\;A^{(1)}_{\nu,0}\,(\alpha),$$

(1.6)
$$v_{\nu} = a_{1}(\nu)B^{(1)}_{\nu,1}\,(\beta)\;A^{(1)}_{\nu,1}\,(\alpha),$$

$$w_{\nu} = a_{2}(\nu)\beta^{(1)}_{\nu,2}\,(\beta)\;A^{(1)}_{\nu,2}\,(\alpha),$$

where $A^{(1)}_{\nu,m}(\alpha)$ and $B^{(1)}_{\nu,m}(\beta)$ are the wave paraboloid functions connected with the hypergeometric function by means of the relations

$$A^{(1)}_{\nu,m}(\alpha) = (ik\alpha^{2})^{\frac{m}{2}}\,e^{-\frac{ik\alpha^{2}}{2}}\,F(\nu,m + 1,\ ik\alpha^{2}) \equiv (ik\alpha^{2})^{-1/2}M_{-\nu+(m+1)/2,m/2}(ik\alpha^{2}),$$

(1.7)
$$B^{(1)}_{\nu,m}(\beta) = (ik\beta^{2})^{\frac{m}{2}}e^{-\frac{ik\beta^{2}}{2}}\,F(m+1-\nu,m+1,ik\beta^{2}) \equiv (ik\beta^{2})^{-1/2}M_{\nu-(m+1)/2,m/2}(ik\beta^{2})$$

and satisfying, respectively, the equations

$$(1.8)\quad \begin{aligned} &A'' + \frac{1}{\alpha} A' + \left[k^2\alpha^2 - \frac{m^2}{\alpha^2} + 4ik\left(\frac{m+1}{2} - \nu\right)\right]A = 0, \\ &B'' + \frac{1}{\beta} B' + \left[k^2\beta^2 - \frac{m^2}{\beta^2} - 4ik\left(\frac{m+1}{2} - \nu\right)\right]B = 0. \end{aligned}$$

Then the secondary field constructed out of the special solutions (1.6) is of the form

$$(1.9)\quad \begin{aligned} u^1 &= \int_{L_o} a_o(\nu)\, B^{(1)}_{\nu,o}(\beta)\, A^{(1)}_{\nu,o}(\alpha)\, d\nu \qquad 0 < R(\nu) < \tfrac{1}{2}\ , \\ v^1 &= \int_{L_1} a_1(\nu)\, B^{(1)}_{\nu,1}(\beta)\, A^{(1)}_{\nu,1}(\alpha)\, d\nu \qquad \tfrac{1}{2} < R(\nu) < 1\ , \\ w^1 &= \int_{L_2} a_2(\nu)\, B^{(1)}_{\nu,2}(\beta)\, A^{(1)}_{\nu,2}(\alpha)\, d\nu \qquad 1 < R(\nu) < \tfrac{3}{2}\ , \end{aligned}$$

where the paths L_o, L_1, L_2 are parallel to the imaginary axis in the plane of the complex variable ν (Fig. 2).

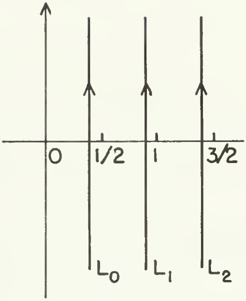

Figure 2

The choice of paths of integration is conditioned by the fact that the special solutions of the type (1.6) must satisfy definite conditions for $\alpha \to \infty$, namely, they must decrease or, at least, not increase as $\alpha \to \infty$ and must correspond to a wave going out to infinity.* The unknown functions $a_i(\nu)$ must be found from the two boundary conditions (1.4) and the condition div $E^1 = 0$.

2. Determination of the coefficients of the representations

In order to find the values of the coefficients $a_i(\nu)$ in the representations of the components of the secondary field it is necessary to make certain assumptions concerning the form of the required functions. The correctness of these assumptions can be tested after we shall have obtained explicit expressions for those functions. Let us put

$$(2.1) \qquad a_o(\nu) = -\frac{pk^2}{c\pi i} e^{-i\pi\nu} \Gamma(\nu) \Gamma(1-\nu)\, c_o(\nu),$$

$$a_1(\nu) = \frac{2pk^2}{c\pi i} e^{-i\pi\nu} \Gamma(\nu) \Gamma(2-\nu)\, c_1(\nu),$$

$$a_2(\nu) = \frac{pk^2}{4c\pi i} e^{-i\pi\nu} \Gamma(\nu) \Gamma(3-\nu)\, c_2(\nu),$$

where we suppose the functions $c_i(\nu)$ regular in the appropriate strips of the plane of the complex variable $\nu = \sigma + i\tau$. Thus, $c_o(\nu)$ is supposed regular (at least) in the strip $0 < R(\nu) < 1$, $c_1(\nu)$ (at least) in the strip $-1 < R(\nu) < 2$, and $c_2(\nu)$ (at least) in the strip $0 < R(\nu) < 2$. In addition we assume that in these strips the coefficients $c_i(\nu)$ are of the order $\mathcal{O}(\tau^2 e^{-2\sqrt{k\tau}\,\beta_o})$ for $\tau \to \infty$ and of the order $\mathcal{O}(|\tau^2|$ for $\tau \to -\infty$.

We now turn to the condition div $E^1 = 0$. If we take into consideration

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[*] One should keep in mind the asymptotic behavior of the functions $A^{(1)}_{\nu,m}(\alpha)$ for $\alpha \to \infty$:

$$A^{(1)}_{\nu,m}(\alpha) \underset{\alpha\to\infty}{\approx} \frac{\Gamma(m+1)}{\Gamma(\nu)} (ik\alpha^2)^{\nu-1-\frac{m}{2}} e^{\frac{ik\alpha^2}{2}} \left[1+\mathcal{O}(\alpha^{-2})\right] +$$

$$+ \frac{\Gamma(m+1)}{\Gamma(m+1-\nu)} (ik\alpha^2)^{\frac{m}{2}-\nu} e^{-\frac{ik\alpha^2}{2}} e^{i\pi\nu}\left[1+\mathcal{O}(\alpha^{-2})\right].$$

the relations connecting derivatives in the paraboloid system of coordinates and the Cartesian system of coordinates, namely

$$\frac{\partial}{\partial x} = \frac{\cos \phi}{\alpha^2+\beta^2}\left(\beta \frac{\partial}{\partial \alpha} + \alpha \frac{\partial}{\partial \beta}\right) - \frac{\sin \phi}{\alpha\beta}\frac{\partial}{\partial \phi},$$

$$(2.2)\quad \frac{\partial}{\partial y} = \frac{\sin \phi}{\alpha^2+\beta^2}\left(\beta \frac{\partial}{\partial \alpha} + \alpha \frac{\partial}{\partial \beta}\right) + \frac{\cos \phi}{\alpha\beta}\frac{\partial}{\partial \phi},$$

$$\frac{\partial}{\partial z} = \frac{1}{\alpha^2+\beta^2}\left(\alpha \frac{\partial}{\partial \alpha} - \beta \frac{\partial}{\partial \beta}\right),$$

we arrive, on the basis of the formulas (1.9) and the condition div $E^1 = 0$, at the following equality:

$$\begin{aligned}
&\int_{L_0} a_0(\nu)\left[\frac{\beta}{\alpha^2+\beta^2} B^{(1)}_{\nu,0}(\beta)\, A^{(1)\prime}_{\nu,0}(\alpha) + \frac{\alpha}{\alpha^2+\beta^2} B^{(1)\prime}_{\nu,0}(\beta)\, A^{(1)}_{\nu,0}(\alpha)\right] d\nu \,+ \\
&+ \int_{L_1} a_1(\nu)\left[\frac{\alpha}{\alpha^2+\beta^2} B^{(1)}_{\nu,1}(\beta) A^{(1)\prime}_{\nu,1}(\alpha) - \frac{\beta}{\alpha^2+\beta^2} B^{(1)\prime}_{\nu,1}(\beta) A^{(1)}_{\nu,1}(\alpha)\right] d\nu \,+ \\
(2.3)\quad &+ \int_{L_2} a_2(\nu)\left[\frac{\beta}{\alpha^2+\beta^2} B^{(1)}_{\nu,2}(\beta) A^{(1)\prime}_{\nu,2}(\alpha) + \frac{\alpha}{\alpha^2+\beta^2} B^{(1)\prime}_{\nu,2}(\beta) A^{(1)}_{\nu,2}(\alpha) \,+\right. \\
&\qquad \left. + \frac{2}{\alpha\beta} B^{(1)}_{\nu,2}(\beta)\, A^{(1)}_{\nu,2}(\alpha)\right] d\nu = 0.
\end{aligned}$$

To deduce from this equation a connection between the coefficients it is necessary to represent the integrals in (2.3) as integrals in the same wave paraboloid functions. Noting that the integrals under consideration are odd functions of the variable α we use for this purpose the functions $A^{(1)}_{\nu,1}(\alpha)$. This leads to the following integral equation for the required coefficients $c_i(\nu)$:

(2.4)
$$\int_{L_0} \frac{c_0(\nu)e^{-i\pi\nu}}{\sin \pi(\mu - \nu)}\, d\nu - 2\int_{L_1} \frac{c_1(\nu)e^{-i\pi\nu}}{\sin \pi(\mu-\nu)}\, d\nu + 2ie^{-i\pi\mu}\, c_1(\mu) + \int_{L_2} \frac{c_2(\nu)e^{-i\nu\pi}}{\sin \pi(\mu-\nu)}\, d\nu = 0 \qquad \left(\tfrac{1}{2} < R(\mu) < 1\right)^{*}.$$

If we write

$$2ie^{-i\pi\mu}\, c_1(\mu) = \int_{L_1} \frac{c_1(\nu)e^{-i\pi\nu}}{\sin \pi(\mu-\nu)}\, d\nu - \int_{L_2} \frac{c_1(\nu)e^{-i\pi\nu}}{\sin \pi(\mu-\nu)}\, d\nu ,$$

then, using Cauchy's theorem, we can write (2.4) as a homogeneous integral equation with the kernel cosec $\pi(\mu-\nu)$. This leads to the following functional relation for the coefficients:

(2.5)
$$c_0(\nu) - c_1(\nu) - c_1(\nu+1) + c_2(\nu+1) = 0.$$

To make use of the boundary conditions (1.4) for the derivation of other connections among the unknown functions $c_i(\nu)$ it is necessary to represent the primary field using integrals in the wave paraboloid functions. Applying the inversion formula[**]

(2.6)
$$f(\alpha) = \frac{k}{\pi\Gamma^2(m+1)} \int_{L_m} e^{-i\pi\nu}\, \Gamma(\nu)\Gamma(m+1-\nu)\bar{f}\, A^{(1)}_{\nu,m}(\alpha)\, d\nu,$$
$$\bar{f} = \int_0^{\infty} \alpha f(\alpha) A^{(1)}_{\nu,m}(\alpha)\, d\alpha \qquad \left(\tfrac{m}{2} < R(\nu) < \tfrac{m+1}{2}\right) ,$$

---

[*] It is also assumed that μ is located to the right of L_1.

[**] A formal derivation of the relations (2.6) is based on the possibility of representing the integral transform under consideration as the result of repeated application of the simpler Mellin and Hankel transforms.

to the functions (1.5) which represent the field of the dipole, we obtain the representations

$$(2.7)\qquad \begin{aligned} u^o &= \int_{L_o} a_o^o(\nu) B_{\nu,o}^{(3)}(\beta)\ A_{\nu,o}^{(1)}(\alpha)\ d\nu && \left(0 < R(\nu) < \tfrac{1}{2}\right), \\ v^o &= \int_{L_1} a_1^o(\nu) B_{\nu,1}^{(3)}(\beta)\ A_{\nu,1}^{(1)}(\alpha)\ d\nu && \left(\tfrac{1}{2} < R(\nu) < 1\right), \\ w^o &= \int_{L_2} a_2^o(\nu) B_{\nu,2}^{(3)}(\beta)\ A_{\nu,2}^{(1)}(\alpha)\ d\nu && \left(1 < R(\nu) < \tfrac{3}{2}\right), \end{aligned}$$

where

$$(2,8)\qquad \begin{aligned} a_o^o(\nu) &= \frac{pk^2}{c\pi i}\, e^{-i\pi\nu}(\nu^2-\nu+1)\ \Gamma(\nu)\Gamma^2(1-\nu), \\ a_1^o(\nu) &= \frac{2pk^2}{c\pi i}\, e^{-i\pi\nu}(1-\nu)\ \Gamma(\nu)\ \Gamma^2(2-\nu), \\ a_2^o(\nu) &= -\frac{pk^2}{2c\pi i}\, e^{-i\pi\nu}\ \Gamma(\nu)\ \Gamma^2(3-\nu), \end{aligned}$$

and where $B_{\nu,m}^{(3)}(\beta)$ are the wave paraboloid functions of the third kind connected with Whittaker's function by means of the relation

$$(2.9)\qquad B_{\nu,m}^{(3)}(\beta) = (ik\beta^2)^{-1/2}\ W_{\nu - \frac{m+1}{2},\frac{m}{2}}(ik\beta^2).$$

In the case under consideration we can represent both sides of (1.4) in the same paraboloid functions using the recursive formulas

$$(2.10)\qquad \begin{aligned} \sqrt{ik\alpha}\ A_{\nu,m}^{(1)}(\alpha) &= \frac{1}{m+1}\left[\nu A_{\nu+1,m+1}^{(1)}(\alpha) + (m+1-\nu)A_{\nu,m+1}^{(1)}(\alpha)\right], \\ \sqrt{ik\alpha}\ A_{\nu,m+1}^{(1)}(\alpha) &= (m+1)\left[A_{\nu,m}^{(1)}(\alpha) - A_{\nu-1,m}^{(1)}(\alpha)\right], \end{aligned}$$

for the functions $A^{(1)}_{\nu,m}(\alpha)$, which leads to the following relations among the required coefficients:

$$
\begin{gathered}
\sqrt{ik\beta_o}\left[c_o(\nu-1)B^{(1)}_{\nu-1,o}(\beta_o) - c_o(\nu)B^{(1)}_{\nu,o}(\beta_o)\right] - \frac{\sqrt{ik\beta_o}}{2(1-\nu)}\Big[c_2(\nu)(1-\nu)(2-\nu)B^{(1)}_{\nu,2}(\beta_o) \\
+ c_2(\nu+1)\nu(1-\nu)B^{(1)}_{\nu+1,2}(\beta_o)\Big] = -\Gamma(1-\nu)B^{(3)}_{\nu+1,1}(\beta_o) - \Gamma(3-\nu)B^{(3)}_{\nu-1,1}(\beta_o),
\end{gathered}
\tag{2.11}
$$

$$
\begin{aligned}
c_o(\nu)\sqrt{ik\beta_o}\,B^{(1)}_{\nu,o}(\beta_o) = {}& c_1(\nu)(1-\nu)B^{(1)}_{\nu,1}(\beta_o) + c_1(\nu-1)\nu B^{(1)}_{\nu+1,1}(\beta_o) \\
& - \Gamma(2-\nu)\nu B^{(3)}_{\nu,1}(\beta_o) + \Gamma(2-\nu)B^{(3)}_{\nu+1,1}(\beta_o).
\end{aligned}
$$

Making use of (2.5), (2.11), and function theoretical considerations, we arrive at the equation

$$
c_1(\nu) - c_1(\nu-1) = f(\nu), \tag{2.12}
$$

where

$$
f(\nu) = \frac{2+(3-2\nu)\ \Gamma(2-\nu)\left[(2-\nu)B^{(3)}_{\nu-1,1}(\beta_o)B^{(1)}_{\nu-1,1}(\beta_o) - B^{(3)}_{\nu,1}(\beta_o)B^{(1)}_{\nu,1}(\beta_o)\right]}{(2-\nu)B^{(1)^2}_{\nu-1,1}(\beta_o) - (1-\nu)B^{(1)^2}_{\nu,1}(\beta_o)}\ . \tag{2.13}
$$

We note that the function $f(\nu)$ is regular at least for $R(\nu) < 2$, since both numerator and denominator are entire functions in the part of the plane under consideration. In addition the expression in the denominator does not vanish. If we were to assume the opposite, then, as can be shown, the corresponding homogeneous problem would have a non-trivial solution; a conclusion which contradicts the uniqueness of the solution for the class of problems under consideration.

The solution of equation (2.12) is of the form*

$$c_1(\nu) = \sum_{n=o}^{\infty} f(\nu-n). \tag{2.14}$$

Using (2.5) and (2.11) we obtain

$$c_o(\nu) = c_1(\nu) + \frac{1-\nu\Gamma(1-\nu)\left[(1-\nu)B^{(3)}_{\nu,1}(\beta_o)B^{(1)}_{\nu,1}(\beta_o) - B^{(3)}_{\nu+1,1}(\beta_o)B^{(1)}_{\nu+1,1}(\beta_o)\right]}{(1-\nu)B^{(1)^2}_{\nu,1}(\beta_o) + \nu B^{(1)^2}_{\nu+1,1}(\beta_o)},$$

(2.15)

$$c_2(\nu) = c_1(\nu) - \frac{1+(1-\nu)\Gamma(2-\nu)\left[(2-\nu)B^{(3)}_{\nu-1,1}(\beta_o)B^{(1)}_{\nu-1,1}(\beta_o) - B^{(3)}_{\nu,1}(\beta_o)B^{(1)}_{\nu,1}(\beta_o)\right]}{(2-\nu)B^{(1)^2}_{\nu-1,1}(\beta_o) - (1-\nu)B^{(1)^2}_{\nu,1}(\beta_o)}.$$

3. Investigation of the solution. Transition to geometrical optics.

In view of the results of Section 2, the final expressions for the components of the secondary field are:

$$E^1_x = -\frac{pk^2}{c\pi i}\int_{L_o} c_o(\nu)e^{-i\pi\nu}\,\Gamma(\nu)\Gamma(1-\nu)B^{(1)}_{\nu,o}(\beta)\;A^{(1)}_{\nu,o}(\alpha)\,d\nu$$

$$+\frac{pk^2}{4c\pi i}\int_{L_2} c_2(\nu)e^{-i\pi\nu}\Gamma(\nu)\Gamma(3-\nu)B^{(1)}_{\nu,2}(\beta)\;A^{(1)}_{\nu,2}(\alpha)\,d\nu\,\cos 2\phi,$$

(3.1)

$$E^1_y = \frac{pk^2}{4c\pi i}\int_{L_2} c_2(\nu)e^{-i\pi\nu}\Gamma(\nu)\Gamma(3-\nu)B^{(1)}_{\nu,2}(\beta)\;A^{(1)}_{\nu,2}(\alpha)\,d\nu\,\sin 2\phi,$$

$$E^1_z = \frac{2pk^2}{c\pi i}\int_{L_1} c_1(\nu)e^{-i\pi\nu}\Gamma(\nu)\Gamma(2-\nu)B^{(1)}_{\nu,1}(\beta)\;A^{(1)}_{\nu,1}(\alpha)\,d\nu\,\cos\phi,$$

---

*It should be noted that the solution of (2.12) is defined uniquely to within a periodic function with period one. On the basis of the assumptions about the coefficient $c_1(\nu)$ it can be shown that in the case under consideration this periodic function vanishes identically.

where the coefficients $c_i(\nu)$ are defined by the formulas (2.13) - (2.15).

If we assume $\alpha < \beta$, then it is possible to supplement the paths in the above integrals by arcs of circles of radius $R \to \infty$. Inside the closed contour the integrands are mesomorphic functions with poles at the points $\nu = -m$ $(m = 0,1,2, \ldots)$. Using the residue theorem we find the components of the secondary field in the form of Pinney series. Thus, for example,

$$(3.2) \qquad E_z^1 = \frac{4pk^2}{c} \sum_{m=0}^{\infty} (m+1) B^{(1)}_{-m,1}(\beta) A^{(1)}_{-m,1}(\alpha) \cos \phi \sum_{p=m}^{\infty} f(-p),$$

where the functions $A^{(1)}_{-m,1}(\alpha)$ are connected with the Laguerre polynomials by means of the relations:

$$(3.3) \qquad A^{(1)}_{-m,1}(\alpha) = \sqrt{ik}\alpha \, e^{-\frac{ik\alpha^2}{2}} \frac{L^1_m(ik\alpha^2)}{m+1} \qquad (m = 0,1,2, \ldots).$$

These series converge only in a part of the domain (for $c < p$) so that one cannot without further proof claim that they represent the required solution.*

Our solution of the problem in the form of contour integrals (3.1) can be used for the investigation of the electromagnetic field at any interior point of the paraboloid in many special cases. In this paper we shall dwell on the limiting case of very short waves when $ka \to \infty$ (a is the distance from the vertex to the focus of the paraboloid).

In view of the asymptotic representations of the wave paraboloid functions for large values of the argument, the coefficients $c_i(\nu)$ have the following form as $k \to \infty$:

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*Our results show that the Pinney series represent the solution in that part of the interior of the paraboloid in which they converge.

$$c_0(\nu)e^{-i\pi\nu} \underset{k\to\infty}{\approx} \frac{\Gamma(2-\nu)\Gamma(1-\nu)}{ik\beta_0^2}(k\beta_0^2)^{2\nu} e^{-ik\beta_0^2},$$

$$(3.4) \qquad c_1(\nu)e^{-i\pi\nu} \underset{k\to\infty}{\approx} \frac{2\Gamma(3-\nu)\Gamma(2-\nu)}{(ik\beta_0^2)^3}(k\beta_0^2)^{2\nu} e^{-ik\beta_0^2},$$

$$c_2(\nu)e^{-i\pi\nu} \underset{k\to\infty}{\approx} \frac{\Gamma(3-\nu)\Gamma(2-\nu)}{(ik\beta_0^2)^3}(k\beta_0^2)^{2\nu} e^{-ik\beta_0^2}.$$

Utilizing the integral representation of the product of two wave paraboloid functions[4]

$$B^{(1)}_{\nu,m}(\beta)A^{(1)}_{\nu,m}(\alpha) = \frac{2\Gamma^2(m+1)}{\Gamma(\nu)\Gamma(m+1-\nu)} e^{\frac{i\pi m}{2}} \int_0^{\pi/2} e^{-\frac{ik}{2}(\alpha^2-\beta^2)\cos 2\theta} (\text{tg}\,\theta)^{2\nu-(m+1)}$$

$$\times \quad J_m(k\alpha\beta \sin 2\theta)d\theta, \qquad (0 < R(\nu) < m+1),$$

and the asymptotic formulas (3.4) for the coefficients, we can transform the integrals (3.1). In this way we obtain for, say, the component E_z^1 the relation:

$$E_z^1 \underset{k\to\infty}{\approx} -\frac{8p}{c\pi i}\frac{e^{-ik\beta_0^2}}{k\beta_0^6}\int_0^{\pi/2} e^{-\frac{ik}{2}(\alpha^2-\beta^2)\cos 2\theta} J_1(k\alpha\beta \sin 2\theta)\frac{d\theta}{\text{tg}^2\theta}$$

$$\times \int_{L_1}(2-\nu)\Gamma^2(2-\nu)(k\beta_0^2\text{tg}\,\theta)^{2\nu}\,d\nu \cos\phi.$$

In view of

$$\int_{L_1}(2-\nu)\Gamma^2(2-\nu)(k\beta_0^2\text{tg}\,\theta)^{2\nu}\,d\nu = 4\pi i k^5\beta_0^{10}\text{tg}^5\theta\, K_1(2k\beta_0^2\text{tg}\,\theta),$$

where $K_1(x)$ is the McDonald function, we obtain, upon putting $tg\theta = x$,

$$E_z^1 \underset{k\to\infty}{\approx} -\frac{32p}{c} k^4 \beta_o^4 e^{-ik\beta_o^2} \int_o^\infty e^{-\frac{ik}{2}(\alpha^2-\beta^2)\frac{1-x^2}{1+x^2}} J_1\left(\frac{2k\alpha\beta x}{1+x^2}\right) K_1(2k\beta_o^2 x) \frac{x^3dx}{1+x^2} \cos \emptyset .$$

If we introduce a new variable $2k\beta_o^2 x = t$ and note that for $k \to \infty$ the principal part of the integral corresponds to values of x close to zero, then we obtain, approximately,

$$E_z^1 \underset{k\to\infty}{\approx} -\frac{2p}{c} \frac{e^{-\frac{ik}{2}(\alpha^2-\beta^2)}}{\beta_o^4} e^{-ik\beta_o^2} \cos \emptyset \int_o^\infty K_1(t)\, J_1\left(\frac{\alpha\beta}{\beta_o^2} t\right) t^3 dt,$$

or, noting that (cf. [3] for example)

$$\int_o^\infty K_\mu(at) J_\nu(bt) t^{\mu+\nu+1} dt = \frac{(2a)^\mu (2b)^\nu\, \Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}} ,$$

we have

$$E_z^1 \underset{k\to\infty}{\approx} -\frac{16p}{c} e^{-\frac{ik}{2}(\alpha^2-\beta^2)} \frac{e^{-ik\beta_o^2}}{\beta_o^4} \frac{\alpha\beta}{\beta_o^2} \frac{\cos \emptyset}{\left[1+\left(\frac{\alpha\beta}{\beta_o^2}\right)^2\right]^3} .$$

Proceeding in an analogous manner in the case of the remaining components of the secondary field, we obtain, under the assumption that $ka \to \infty$, the following asymptotic formulas:

$$E^1_x \underset{k\to\infty}{\approx} -\frac{2pk}{ci}\,\frac{e^{-ik(z+2a)}}{2a}\,\frac{1-\left(\frac{\rho}{2a}\right)^2 \cos 2\phi}{\left[1+\left(\frac{\rho}{2a}\right)^2\right]^2}\,,$$

$$(3.5) \qquad E^1_y \underset{k\to\infty}{\approx} -\frac{2pk}{ci}\,\frac{e^{-ik(z+2a)}}{2a}\,\frac{\left(\frac{\rho}{2a}\right)^2 \sin 2\phi}{\left[1+\left(\frac{\rho}{2a}\right)^2\right]^2}\,,$$

$$E^1_z \underset{k\to\infty}{\approx} -\frac{16p}{c}\,\frac{e^{-ik(z+2a)}}{(2a)^2}\,\frac{\frac{\rho}{2a}\cos\phi}{\left[1+\left(\frac{\rho}{2a}\right)^2\right]^2}\,,$$

where we have put $z = \frac{1}{2}(\alpha^2-\beta^2)$, $\rho = \alpha\beta$, $\beta_0^2 = 2a$.

These approximate formulas for the components of the secondary field admit of a simple physical interpretation, since they represent the field of a non-homogeneous plane wave whose phase and amplitude satisfy, as is easily seen, the requirements of geometrical optics. Indeed, the formulas (3.5) show that as $k\to\infty$ the z-component is of a higher order of smallness than the components E^1_x and E^1_y. If we suppose the electrical field known and compute the magnetic field by means of the formula $H = -\frac{1}{ik}\,\mathrm{rot}\,E$, we arrive at the conclusion that here too, as $k\to\infty$, there remain only the x- and y-components of the field, and that

$$H^1_x \underset{k\to\infty}{\approx} -\frac{2pk}{ci}\,\frac{e^{-ik(z+2a)}}{2a}\,\frac{\left(\frac{\rho}{2a}\right)^2 \sin 2\phi}{\left[1+\left(\frac{\rho}{2a}\right)^2\right]^2}\,,$$

$$(3.6)$$

$$H^1_y \underset{k\to\infty}{\approx} -\frac{2pk}{ci}\,\frac{e^{-ik(z+2a)}}{2a}\,\frac{1-\left(\frac{\rho}{2a}\right)^2 \cos 2\phi}{\left[1+\left(\frac{\rho}{2a}\right)^2\right]^2}\,.$$

Thus, in view of the formulas (3.5) and (3.6), the mean value over a period of the only non-trivial component of the Poynting vector, namely, the z-component, is given by

$$(3.7) \qquad d\bar{S}_z = \frac{p^2k^2}{8\pi c a^2}\,\frac{1-2\left(\frac{\rho}{2a}\right)^2\cos 2\phi + \left(\frac{\rho}{2a}\right)^4}{\left[1+\left(\frac{\rho}{2a}\right)^2\right]^4}\,\rho d\rho d\phi\,.$$

On the other hand, if we compute the stream of energy radiated by the dipole and contained in the cone between the angles θ and $\theta+d\theta$ and if we assume that, in accordance with the laws of geometrical optics, this whole stream must, after reflection from the paraboloid, find itself in the cylindrical layer of thickness $d\rho$ (Fig. 3), where $d\rho$ is connected with the angle $d\theta$ by means of the relation

$$d\theta = \frac{d\rho}{a\left[1+\left(\frac{\rho}{2a}\right)^2\right]},$$

we are led to the equality

$$(3,8) \qquad dw = \frac{p^2k^2}{8\pi c}\sin^2\Psi\,\sin\theta\; d\theta d\phi,$$

where Ψ is the angle between r and the x-axis, i.e., $\cos\Psi = \sin\theta\cos\phi$.

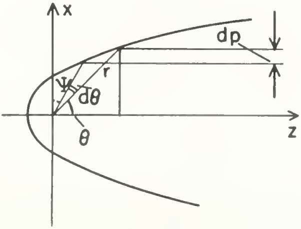

Figure 3

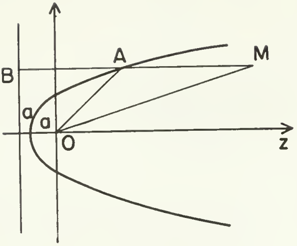

Figure 4

Noting that $\sin\theta = \dfrac{\rho}{a\left[1+\left(\frac{\rho}{2a}\right)^2\right]}$ we obtain the formula

$$(3.9) \qquad dw = \frac{p^2k^2}{8\pi ca^2}\,\frac{1-2\left(\frac{\rho}{2a}\right)^2\cos 2\phi+\left(\frac{\rho}{2a}\right)^4}{\left[1+\left(\frac{\rho}{2a}\right)^2\right]^4}\,\rho\,d\rho\,d\phi\ ,$$

which coincides with the equality (3.7) obtained directly from the asymptotic expressions for the electromagnetic field inside the paraboloid for $k \to \infty$.

The fact that the phase of the secondary field is also in agreement with the laws of geometrical optics is easy to verify. Indeed, if we consider in Fig. 4 the paths of rays emanating from the radiating dipole located at the focus of the paraboloid, then, according to the laws of geometrical optics, the phase of the secondary field at the point M will be $k(OA + AM)$. If we use the geometric relations which obtain, we find that $OA = AB = 2a + z_A$, so that $OA + AM = 2a + z_M$, i.e., the secondary field phase is exactly equal to the phase which enters into the relations (3.5) and (3.6).

In conclusion the author wishes to thank G. A. Grinberg for having

suggested the topic of this paper and to N. N. Lebedev for many helpful suggestions.

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APPENDIX

W. Magnus and N.N. Lebedev

Since it is not obvious how to derive equation (2.4) in this paper, a proof of this equation will be sketched here. We are indebted to Professor N. N. Lebedev of the Physico Technical Institute in Leningrad for communicating this proof to us after consulting with the author of the paper.

The derivation of (2.4) is based on the assumption that a solution of the equation

$$\text{(A1)}\quad \frac{1}{\alpha}\frac{\partial}{\partial\alpha}\alpha\frac{\partial v}{\partial\alpha} + \frac{1}{\beta}\frac{\partial}{\partial\beta}\beta\frac{\partial v}{\partial\beta} + k^2(\alpha^2+\beta^2)v - \left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)v = 0$$

regular within the paraboloid and satisfying the radiation condition

$$\text{(A2)}\quad v = \mathcal{O}(\alpha^{-1}),\quad \frac{1}{\alpha}\frac{\partial v}{\partial\alpha} + ikv = o(\alpha^{-1}) \qquad (\alpha\to\infty)$$

can be expressed in the form of an integral of the type

$$\text{(A3)}\quad v = v(\alpha,\beta) = \int_{L_1} M(\mu)\, B^{(1)}_{\mu,1}(\beta)\, A^{(1)}_{\mu,1}(\alpha)\, d\mu.$$

In order to determine the coefficient $M(\mu)$ let us divide (A3) by $\sqrt{ik}\,\beta$ and pass to the limit $\beta\to 0$.

This gives

$$\text{(A4)}\quad f(\alpha) = \lim_{\beta\to 0}\frac{v(\alpha,\beta)}{\sqrt{ik}\,\beta} = \int_{L_1} M(\mu)\, A^{(1)}_{\mu,1}(\alpha)\, d\mu,$$

from which it follows, in consequence of the inversion formula (2.6), that

$$\text{(A5)}\quad M(\mu) = ke^{-i\pi\mu}\frac{1-\mu}{\sin\pi\mu}\int_0^\infty \alpha\, f(\alpha)\, A^{(1)}_{\mu,1}(\alpha)\, d\alpha.$$

It may be easily proved that the functions in brackets under the integral signs in (2.3), and which will be denoted in what follows by $v_o(\alpha,\beta)$, $v_1(\alpha,\beta)$, $v_2(\alpha,\beta)$, are regular solutions of equation (1). $v_o(\alpha,\beta)$ and $v_2(\alpha,\beta)$ also satisfy condition (A2).

Using this method we get

$$\text{(A6)}\quad v_o(\alpha,\beta) = \frac{\beta}{\alpha^2+\beta^2} B^{(1)}_{\nu,o}(\beta)\, A^{(1)\prime}_{\nu,o}(\alpha) + \frac{\alpha}{\alpha^2+\beta^2} B^{(1)\prime}_{\nu,o}(\beta)\, A^{(1)}_{\nu,o}(\alpha)$$

$$= \int_{L_1} M_o(\mu)\, B^{(1)}_{\mu,1}(\beta)\, A^{(1)}_{\mu,1}(\alpha)\,d\mu,$$

$$f_o(\alpha) = \lim_{\beta\to 0} \frac{v_o(\alpha,\beta)}{\sqrt{ik}\,\beta} = \frac{1}{\sqrt{ik}\,\alpha^2}\left[A^{(1)\prime}_{\nu,o}(\alpha) + (1-2\nu)ik\alpha\, A^{(1)}_{\nu,o}(\alpha)\right]$$

$$= \nu(1-\nu)\frac{\sqrt{ik}}{\alpha}\, A^{(1)}_{\nu+1,2}(\alpha),$$

from which

$$\text{(A7)}\quad M_o(\mu) = \sqrt{ik}\, k\nu(1-\nu)e^{-i\pi\mu}\,\frac{1-\mu}{\sin\pi\mu}\int_o^\infty A^{(1)}_{\nu+1,2}(\alpha)\, A^{(1)}_{\mu,1}(\alpha)\,d\alpha .$$

Taking into consideration that $0 < R(\nu) < \frac{1}{2}$ and $\frac{1}{2} < R(\mu) < 1$ and making use of the integral representations

$$A^{(1)}_{\mu,1}(\alpha) = 2e^{i\pi/4}\,\frac{e^{i\pi\mu/2}}{\Gamma(\mu)}\, e^{-ik\alpha^2/2}\int_o^\infty J_1(2\sqrt{k}\,\alpha t)e^{-it^2}t^{2\mu-2}dt,$$

(A8)

$$A^{(1)}_{\nu+1,2}(\alpha) = -4i\,\frac{e^{i\pi\nu/2}}{\Gamma(2-\nu)}\, e^{ik\alpha^2/2}\int_o^\infty J_2(2\sqrt{k}\,\alpha\tau)e^{i\tau^2}\tau^{1-2\nu}d\tau,$$

we obtain after some calculations which we shall give at the end of our analysis:

$$(A9) \quad \int_0^\infty A^{(1)}_{\nu+1,2}(\alpha)\, A^{(1)}_{\mu,1}(\alpha)\, d\alpha = \frac{\sin \pi\nu}{\sqrt{ik}\, \nu\, (1-\nu)} \; \frac{e^{i\pi\mu}}{\sin \pi(\mu-\nu)} \, .$$

Therefore

$$(A10) \quad M_o(\lambda) = k\, \frac{1-\mu}{\sin \pi\mu} \; \frac{\sin \pi\nu}{\sin \pi(\mu-\nu)} \, .$$

The case of the function $v_2(\alpha,\beta)$ may be treated in a similar way.

The function $v_1(\alpha,\beta)$ does not satisfy condition (A2). But if it is written in the form

$$(A11) \quad v_1(\alpha,\beta) = v_1^*(\alpha,\beta) - ikB^{(1)}_{\nu,1}(\beta)\, A^{(1)}_{\nu,1}(\alpha)$$

the method described above may be applied to $v_1^*(\alpha,\beta)$ where one must put $\frac{1}{2} < R(\nu) < R(\mu) < 1$.

Introducing into (2.3) the expressions obtained for the functions $v_o(\alpha,\beta)$, $v_1(\alpha,\beta)$, $v_2(\alpha,\beta)$ and equating to zero the coefficient of $B^{(1)}_{\mu,1}(\beta)\, A^{(1)}_{\mu,1}(\alpha)$, we get the integral equation (2.4).

We still have to derive equation (A9) from equations (A8). This can be done in the following manner: We have from (A8)

$$\int_0^\infty A^{(1)}_{\nu+1,2}(\alpha)\, A^{(1)}_{\mu,1}(\alpha)\, d\alpha =$$

$$\frac{8e^{-i\pi/4}\, e^{i\pi(\mu+\nu)/2}}{\Gamma(\mu)\Gamma(2-\nu)} \int_0^\infty d\alpha \int_0^\infty\!\!\int_0^\infty J_1(2\sqrt{k}\, \alpha t) J_2(2\sqrt{k}\, \alpha\tau) e^{i(\tau^2-t^2)} t^{2\mu-2} \tau^{1-2\nu}\, dt\, d\tau$$

We shall exchange the order of integration, carrying out the integration with respect to α first. The well known Sonine-Schafheitlin formulas show that

$$\int_0^\infty J_1(2\sqrt{k}\, \alpha t)\, J_2(2\sqrt{k}\, \alpha\, \tau)\, d\alpha = \begin{cases} t\,\tau^{-2} & \text{if } 0 \leq t < \tau \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we find for the left-hand side of (A8) the expression

$$\text{(A12)}\quad \frac{8e^{-i\pi/4+i\pi(\mu+\nu)/2}}{\Gamma(\mu)\Gamma(2-\nu)2\sqrt{k}} \int_0^\infty d\tau \int_0^\tau dt\; e^{i(\tau^2-t^2)} t^{2\mu-1} \tau^{-1-2\nu}.$$

By expanding e^{-it^2} in a power series and integrating term by term with respect to t we find

$$\text{(A13)}\quad \int_0^\tau e^{-it^2}\, t^{2\mu-1}\, dt = \frac{\tau^{2\mu}}{2\mu}\; {}_1F_1\,(\mu;\ \mu+1;\ -i\tau^2).$$

The second integration in (A12) reduces now to an evaluation of the integral

$$\text{(A14)}\quad \frac{1}{2\mu}\int_0^\infty e^{i\tau^2}\tau^{2\mu-2\nu-1}\; {}_1F_1\,(\mu;\mu+1;\ -i\tau^2)d\tau.$$

By changing the path of integration or, equivalently, by substituting a new variable s for $-i\tau^2$, we find for (A14):

$$\frac{\exp[i\pi(\mu-\nu)/2]}{4\mu}\int_0^\infty e^{-s}\, s^{\mu-\nu-1}\; {}_1F_1(\mu;\ \mu+1;\ s)ds$$

$$= \frac{\Gamma(\mu-\nu)\exp[i\pi(\mu-\nu)/2]}{4\mu}\,{}_2F_1\,(\mu-\nu,\mu;\ \mu+1;\ 1)$$

$$= (4\mu)^{-1}\,\frac{\Gamma(\mu-\nu)\Gamma(\mu+1)\Gamma(\nu+1-\mu)}{\Gamma(1)\ \Gamma(\nu+1)}\ \exp[i\pi(\mu-\nu)/2]$$

$$= \frac{\pi}{4}\ \frac{\Gamma(\mu)}{\Gamma(\nu+1)}\ \frac{\exp[i\pi(\mu-\nu)/2]}{\sin\pi(\mu-\nu)}.$$

Combining this result with (A13) and (A12) we find that (A9) is true.

The analysis given is purely formal.

DERIVATION OF THE LAWS OF REFLECTION OF GEOMETRICAL OPTICS ON THE BASIS OF AN ASYMPTOTIC TREATMENT OF THE DIFFRACTION PROBLEM

M.I. Kontorovitch and U.K. Muravév

(Translated by Abe Shenitzer,
with Foreword by M. Kline)

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Foreword

The following paper by Kontorovitch and Muravév is concerned primarily with the asymptotic evaluation of double integrals of the form

$$\int_S Fe^{ik\phi} dS.$$

Because this subject was being investigated by two of our own people and it was desirable that we learn the contents of the Russian paper, a translation was undertaken.

The purpose of the paper is to extend to double integrals the application of the method of stationary phase which in the past had been used to evaluate single integrals. The authors do make such an extension in that they obtain the principal part of the double integral and show that it is of the order of $1/k$. Their result is shown in formulas (9) and (11) of their paper. Actually the term obtained by the authors is only one of a number of terms, some of the same order and others of higher order. These other terms come from other types of stationary points, or critical points as they are called in the literature, but these are not considered by the present authors. They limit themselves specifically to the condition that the phase function ϕ contains at most one stationary point and moreover a stationary point at which the phase function is either a maximum or a minimum.

The results obtained by these authors have now been superseded by the work of a number of others. I should like to refer to our own Research Report No. EM-100 by Douglas S. Jones and the undersigned entitled 'Asymptotic Expansion of Multiple Integrals and the Method of Stationary Phase.' This report, as well as a number of others cited in the introduction and bibliography indicate the more recent work that has been done and the progress made beyond that in the paper by Kontorovitch and Muravév.

Morris Kline

Abstract

In this paper we consider the derivation of an asymptotic formula for surface integrals of oscillating functions (the method of stationary phase). By means of the formula so obtained we establish a connection between the laws of reflection of geometrical optics and the laws of classical electrodynamics taking as our starting point the integral formulation of the diffraction problem.

1. Preliminary remarks

A large number of problems in mathematical physics and, in particular, in radio physics and in optics involves the computation of integrals of fast oscillating functions. The study of such integrals is of interest not only from the point of view of solving concrete problems but also from the point of view of establishing general relations which are valid in the case of very high frequencies.

In particular, it is possible to show that the problem of connecting many of the laws of geometrical optics with classical electrodynamics can be reduced to the problem of studying the behavior of integrals of the type mentioned above in which the integrands are functions with large frequency of oscillation.

For one-dimensional integrals of this type the corresponding asymptotic representations can be obtained using a well-known formula which is a consequence of the principle of stationary phase.

However, it is not possible to justify fully the extension of this formula to the case of integrals taken over a two-dimensional domain and it is precisely domains of this type that usually turn up in the problems mentioned above.

With these considerations in mind we give below the derivation of an asymptotic formula which permits us to obtain approximate expressions for integrals of the type mentioned above. This formula represents an extension of the method of stationary phase to the case of two-dimensional integrals.

Since a surface integral can be reduced to one or more integrals over plain domains, we restrict ourselves to the derivation of an asymptotic formula for a plane domain. The result obtained can be extended to the case of a curved surface.

2. Derivation of the asymptotic formula

Consider the integral

$$I = \int_S F e^{ik\phi} \, dS \tag{1}$$

taken over a plain domain S. Here F and ϕ are real functions of two variables x and y defined in the domain S and on its boundary Γ and k is a constant real number which we shall assume to be large.

We impose the following restrictions[*] on the functions F and ϕ: F is single-valued, continuous and finite together with all its derivatives of first and second order. ϕ is single-valued, continuous and finite together with its partial derivatives up to the fourth order, and the interior of S contains at most one stationary point in which grad ϕ[**] vanishes and

$$\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{dxdy}\right)^2 > 0, \tag{2}$$

---

[*] Obviously sufficient but not necessary.

[**] In the sequel grad and div denote surface differential operators.

i.e., the stationary point corresponds to a minimum or maximum of the function $\emptyset$.

Let S_o be a small circular domain about the stationary point x_o, y_o. Let γ be the boundary of S_o(cf. Fig. 1). We have

$$I = \int_{S_o} Fe^{ik\emptyset}\, dS + \int_{S-S_o} Fe^{ik\emptyset}\, dS.$$

It is easy to obtain the identity

$$Fe^{ik\emptyset} = -\frac{1}{ik}\left[e^{ik\emptyset} \operatorname{div} \frac{F\vec{q}}{(\vec{q}\ \mathrm{grad}\ \emptyset)} - \operatorname{div} \frac{F\vec{q}\, e^{ik\emptyset}}{(\vec{q}\ \mathrm{grad}\ \emptyset)} \right], \tag{3}$$

where $\vec{q}$ is an arbitrary vector.

At first we assume that nowhere in the domain under consideration, except for the stationary point, is grad $\emptyset$ perpendicular to the radius vector $\vec{r}$ connecting the stationary point with a point of S.

Then, putting $\vec{q} = \vec{r}_o$, where $\vec{r}_o = \dfrac{\vec{r}}{|\vec{r}|}$, we can write:

$$I = \int_{S_o} Fe^{ik\emptyset}\, dS - \frac{1}{ik}\left[\int_{S-S_o} e^{ik\emptyset} \operatorname{div} \frac{F\vec{r}_o}{(\vec{r}_o \mathrm{grad}\ \emptyset)}\, dS - \int_{S-S_o} \operatorname{div} \frac{Fe^{ik\emptyset}\, \vec{r}_o}{(\vec{r}_o \mathrm{grad}\ \emptyset)}\, dS \right]$$

or, on the basis of Ostrogradsky's theorem,

$$I = \int_{S_o} Fe^{ik\emptyset} dS - \frac{1}{ik}\left[\int_{S-S_o} e^{ik\emptyset} \operatorname{div} \frac{F\vec{r}_o}{(\vec{r}_o \mathrm{grad}\ \emptyset)}\, dS - \int_{r} \frac{Fe^{ik\emptyset}(\vec{m}\ \vec{r}_o)}{(\vec{r}_o \mathrm{grad}\ \emptyset)}\, dl - \right.$$

$$\left. - \int_{\gamma} \frac{Fe^{ik\emptyset}(\vec{m}_o \vec{r}_o)}{(\vec{r}_o \mathrm{grad}\ \emptyset)}\, dl \right]. \tag{4}$$

Here $\vec{m}$ and $\vec{m}_o$ are unit vectors of the normal to the contours Γ and γ and $\vec{m}_o = -\vec{r}_o$.

Now, let the radius r of the circle γ tend to zero. In view of the boundedness of F the integral over S_o will tend to zero. We then obtain

$$I = -\frac{1}{ik}\left[\int_S e^{ik\phi} \operatorname{div} \frac{F r_o}{(\vec{r}_o \operatorname{grad} \phi)} dS - \int_r \frac{F e^{ik\phi}(\vec{m}\,\vec{r}_o)}{(\vec{r}_o \operatorname{grad} \phi)} dl + \lim_{r \to 0} \int_\gamma \frac{F e^{ik\phi}}{\frac{\partial\phi}{\partial r}} dl\right].$$

Using polar coordinates r and θ with origin at the stationary point and ignoring quantities of order higher than r we can write

$$\frac{\partial\phi}{\partial r} = \left[\alpha_1 \cos^2\theta + 2\alpha_2 \cos\theta \sin\theta + \alpha_3 \sin^2\theta\right] r$$

where

$$\alpha_1 = \left.\frac{\partial^2\phi}{\partial x^2}\right|_{st} ; \quad \alpha_2 = \left.\frac{\partial^2\phi}{dxdy}\right|_{st} ; \quad \alpha_3 = \left.\frac{\partial^2\phi}{\partial y^2}\right|_{st} .$$

Thus

$$\lim_{r \to 0} \int_r \frac{F e^{ik\phi}}{\frac{\partial\phi}{\partial r}} dl = F(x_o, y_o) e^{ik\phi(x_o, y_o)} \int_o^{2\pi} \frac{d\theta}{\alpha_1 \cos^2\theta + 2\alpha_2 \cos\theta \sin\theta + \alpha_3 \sin^2\theta} .$$

Under the assumption (2), i.e., for $\alpha_1\alpha_3 - \alpha_2^2 > 0$, this integral can be computed by means of the substitution $tg\theta = z$ and is equal to

$$\int_o^{2\pi} \frac{d\theta}{\alpha_1 \cos^2\theta + 2\alpha_2 \cos\theta \sin\theta + \alpha_3 \sin^2\theta} = 2\int_{-\infty}^{\infty} \frac{dz}{\alpha_1 + 2\alpha_2 z + \alpha_3 z^2} = \frac{2\pi}{\sqrt{\alpha_1\alpha_3 - \alpha_2^2}} .$$

We thus obtain the following expression for I

$$(5) \quad I = -\frac{1}{ik}\left[\frac{2\pi}{\sqrt{\alpha_1\alpha_3 - \alpha_2^2}}\, F(x_o,y_o)e^{ik\phi(x_o,y_o)} - \int_\Gamma \frac{Fe^{ik\phi}(\vec{m}\,\vec{r}_o)}{(\vec{r}_o \operatorname{grad}\phi)}\,dl + \right.$$

$$\left. + \int_S e^{ik\phi}\operatorname{div}\frac{F\vec{r}_o}{(\vec{r}_o\operatorname{grad}\phi)}\,dS\right].$$

This implies

$$(6) \quad \left| ik\, I + \frac{2\pi}{\sqrt{\alpha_1\alpha_3 - \alpha_2^2}}\, F(x_o,y_o)e^{ik\phi(x_o,y_o)} - \int_\Gamma \frac{Fe^{ik\phi}(\vec{m}\,\vec{r}_o)}{(\vec{r}_o\operatorname{grad}\phi)}\,dl \right| =$$

$$= \left| \int_S e^{ik\phi}\operatorname{div}\frac{F\vec{r}_o}{(\vec{r}_o\operatorname{grad}\phi)}\,dS \right|.$$

We now consider the integral on the right side of (6)

$$I_1 = \int_{S_o} e^{ik\phi}\operatorname{div}\frac{F\vec{r}_o}{(\vec{r}_o\operatorname{grad}\phi)}\,dS + \int_{S-S_o} e^{ik\phi}\operatorname{div}\frac{F\vec{r}_o}{(\vec{r}_o\ \operatorname{grad}\phi)}\,dS = I' + I''$$

where S_o is a circular domain of small radius r, about x_o,y_o.

Writing the integral I' in the form

$$I' = \int_{S_o} e^{ik\phi}\operatorname{div}\frac{F\vec{r}_o}{(\vec{r}_o\operatorname{grad}\phi)}\,dS = \int_0^{2\pi}\int_0^{r_1} e^{ik\phi}\frac{\partial}{\partial r}\left(\frac{Fr}{\frac{\partial\phi}{\partial r}}\right)dr\,d\theta,$$

we shall show that the integrand in I' is finite in the whole domain of integration. We can write

$$\frac{\partial}{\partial r}\left(\frac{rF}{\frac{\partial \phi}{\partial r}}\right) = \frac{r}{\frac{\partial \phi}{\partial r}}\frac{\partial F}{\partial r} + F\,\frac{\frac{\partial \phi}{\partial r} - r\frac{\partial^2 \phi}{\partial r^2}}{\left(\frac{\partial \phi}{\partial r}\right)^2} .$$

For small r

$$\frac{\partial \phi}{\partial r} = (\alpha_1\cos^2\theta + 2\alpha_2\cos\theta\sin\theta + \alpha_3\sin^2\theta)r + O(r^2)$$

so that, in accordance with the inequality (2), the first term remains finite for all θ.

As for the second term, we can write

$$\frac{\partial \phi}{\partial r} - r\frac{\partial^2 \phi}{\partial r^2} = r\frac{\partial^2 \phi_o}{\partial r^2} + \frac{r^2}{2}\frac{\partial^3 \phi_o}{\partial r^3} - r\frac{\partial^2 \phi_o}{\partial r^2} - r\frac{\partial^3 \phi_o}{\partial r^3} + O(r^3) = -\frac{r^2}{2}\frac{\partial^2 \phi_o}{\partial r^2} + O(r^3),$$

where the index 'o' means that the quantities involved are computed at x_o, y_o.

It is now easy to see that $\frac{\partial}{\partial r}\left(\frac{rF}{\frac{\partial \phi}{\partial r}}\right)$ is bounded in all of the domain S.

We can therefore write

$$|I'| \leq \int_o^{2\pi} d\theta \int_o^{r_1} \left|\frac{\partial}{\partial r}\;\frac{Fr}{\frac{\partial \phi}{\partial r}}\right| dr.$$

If we now denote by M the largest value of the integrand in the domain of integration, then

$$(7) \qquad |I'| \leq 2\pi r_1 M$$

and M is independent of k and r_1.

We now transform I'' using identity (3) and replacing F in (3) by F_1,

where

$$F_1 = \operatorname{div} \frac{F\vec{r}_o}{(\vec{r}_o \operatorname{grad} \phi)} .$$

Then

$$I'' = -\frac{1}{ik}\left[\int_{S-S_o} e^{ik\phi} \operatorname{div} \frac{F_1\vec{r}_o}{(\vec{r}_o \operatorname{grad} \phi)}\, dS - \int_{\Gamma+\gamma} \frac{F_1 e^{ik\phi}(\vec{m}\,\vec{r}_o)}{(\vec{r}_o \operatorname{grad} \phi)}\, dl\right].$$

Since the integrand remains finite in the domain and along the paths of integration, we may assume that

$$(8) \qquad |I''| < \frac{N}{k}$$

where N is a constant which is independent of k but dependent on r_1.

Given $\varepsilon > 0$ we can make $|I'| < \varepsilon/2$ by choosing r_1 small enough.

Now, keeping r_1 fixed in accordance with (8) and choosing k large enough we can make $|I''| < \varepsilon/2$.

Thus, for a sufficiently large k, the right side of (6) can be made arbitrarily small.

Hence the formula:

$$(9) \quad \int_S Fe^{ik\phi}\, dS = -\frac{1}{ik}\left[\frac{2\pi}{\sqrt{\alpha_1\alpha_3-\alpha_2^2}} F(x_o,y_o)e^{ik\phi(x_1,y_o)} - \int_{\Gamma} \frac{Fe^{ik\phi}(\vec{m}\,\vec{r}_o)}{(\vec{r}_o \operatorname{grad} \phi)} dl\right] +$$

$$+ \frac{1}{k}\, O\left(\frac{1}{k}\right).$$

In the above proof we assumed that, with the exception of points x_o,y_o, the expression $(\vec{r}_o \operatorname{grad} \phi)$ does not vanish in the domain of integration. We shall now try to remove this restriction.

In view of the inequality (2) we can always find in the vicinity of points x_o, y_o a finite circular domain σ with boundary γ_1 in which $(\vec{r}_o \operatorname{grad} \phi) \neq 0$. Then, using (9), we can write

$$I_2 = \int_{\sigma} F e^{ik\phi} dS = -\frac{1}{ik}\left[\frac{2\pi}{\sqrt{\alpha_1\alpha_3-\alpha_2^2}} F(x_o,y_o) e^{ik\phi(x_o,y_o)} - \int_{\gamma_1} \frac{F e^{ik\phi}(\vec{m}_1\vec{r}_o)}{(\vec{r}_o \operatorname{grad} \phi)} dl\right] +$$

$$+ \frac{1}{k} O\left(\frac{1}{k}\right),$$

where $\vec{m}_1$ is the outer normal to γ_1.

We now consider the remaining part of the domain of integration $S-S_o$ bounded by the contours Γ and γ_1 and not containing stationary points. Next we choose a vector $\vec{q}$ (which appears in formula (3)) in such a way that $\vec{q}$ is not perpendicular to grad ϕ on γ_1 and is equal to $\vec{r}_o$ on γ_1*.

Before we continue our argument we show that such a vector can at all be constructed.

We choose $\vec{q} = F_1\vec{r}_o + F_2 \operatorname{grad} \phi$, where F_1 and F_2 are functions of x and y such that $F_1 = 1$ and $F_2 = 0$ on γ_1. Then $(\vec{q} \operatorname{grad} \phi) = F_1 \frac{\partial \phi}{\partial r} + F_2(\operatorname{grad} \phi)^2$ and $\vec{q} = \vec{r}_o$ on γ_1. In view of (2), $\frac{\partial \phi}{\partial r}$ does not change sign inside σ. Hence the nearest point at which $\frac{\partial \phi}{\partial r} = 0$ lies outside the contour γ_1.

We can thus select a domain σ_1 with boundary γ_2 containing the domain σ such that $\frac{\partial \phi}{\partial r}$ does not change sign in σ_1. We assume that F_2 is positive everywhere (except for the contour γ_1) and that F_1 has the same sign as $\frac{\partial \phi}{\partial r}$ has in the vicinity of the stationary point.

---

*We assume that $\vec{q}$ satisfies the same conditions which the conditions imposed on ϕ imply for grad ϕ.

In the interior of the domain contained between the contours γ_1 and γ_2 we have

$$(\vec{q}\ \mathrm{grad}\ \phi) = F_2(\mathrm{grad}\ \phi)^2 + F_1 \frac{\partial\phi}{\partial r} > 0.$$

In the domain exterior to γ_2 we have

$$|(\vec{q}\ \mathrm{grad}\ \phi)| \geq F_2(\mathrm{grad}\ \phi)^2 - \left|F_1 \frac{\partial\phi}{\partial r}\right| .$$

Since in the domain under consideration F_2 and grad ϕ do not vanish, it is always possible to choose F_2 so that $F_2(\mathrm{grad}\ \phi) > \left|F_1 \frac{\partial\phi}{\partial r}\right|$.* Consequently, $(\vec{q}\ \mathrm{grad}\ \phi) > 0$, which is what we set out to prove.

Now, using (3) and the theorem of Ostrogradski we can write:

$$I_3 = \int_{S-\sigma} Fe^{ik\phi} dS = -\frac{1}{ik}\left[\int_{S-\sigma} e^{ik\phi} \mathrm{div}\, \frac{F\vec{q}}{(\vec{q}\ \mathrm{grad}\ \phi)}\, dS - \int_{\Gamma+\gamma_1} \frac{Fe^{ik\phi}(\vec{m}\ \vec{q})}{(\vec{q}\ \mathrm{grad}\ \phi)}\, dl\right].$$

As above, applying formula (3) once more, we obtain

$$\int_{S-\sigma} e^{ik\phi} \mathrm{div}\, \frac{F\vec{q}}{(\vec{q}\ \mathrm{grad}\ \phi)}\, dS = O(\tfrac{1}{k}).$$

We thus obtain the following representation for the integral in question:

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*One way of doing this is to put $F_2 = M(r-r_o)$, r = distance from the stationary point, M = a sufficiently large positive number, r_o = radius of the circle γ_1, and to put $F_1 = 1$.

$$(10)\quad \int_S Fe^{ik\emptyset}dS = I_1 + I_2 = -\frac{1}{ik}\left[\frac{2\pi}{\sqrt{\alpha_1\alpha_3-\alpha_2^2}}F(x_0,y_0)e^{ik\emptyset(x_0,y_0)} - \right.$$

$$\left. -\int_\Gamma \frac{Fe^{ik\emptyset}(\vec{m}\,\vec{q})}{(\vec{q}\ \mathrm{grad}\ \emptyset)}dl\right] + \frac{1}{k}O(\frac{1}{k}).$$

Depending on the definition of $\vec{q}$ on the boundary of the domain we can obtain different representations of the integral under consideration.

In particular, if we put $\vec{q} = \mathrm{grad}\ \emptyset$ in the contour integral, then we find:

$$(11)\quad \int_S Fe^{ik\emptyset}dS = -\frac{1}{ik}\left[\frac{2\pi}{\sqrt{\alpha_1\alpha_3-\alpha_2^2}}F(x_0,y_0)e^{ik\emptyset(x_0,y_0)} - \int_\Gamma \frac{Fe^{ik\emptyset}\frac{\partial\emptyset}{\partial m}}{(\mathrm{grad}\ \emptyset)^2}dl\right] +$$

$$+\frac{1}{k}O(\frac{1}{k}).$$

If we assume that $\vec{q} = \vec{r}_0$ on Γ, then we again obtain formula (9).

It is easy to see that if $\emptyset$ is constant on the whole contour or on each distinct part of the path, then the contour integral in the formulas (9), (10), and (11) is of the same order with respect to k as the first summand.

In the remaining cases this integral will, in general, tend to zero with increasing k.

It is easy to extend formula (9) to the case when the surface is curved and is characterized by Gaussian coordinates u, v, and when we are given the values of the second derivatives of the function $\emptyset$ with respect to these coordinates at the stationary point:

$$\alpha_1 = \left.\frac{\partial^2\phi}{\partial u^2}\right|_{st} ; \qquad \alpha_2 = \left.\frac{\partial^2\phi}{\partial u\partial v}\right|_{st} ; \qquad \alpha_3 = \left.\frac{\partial^2\phi}{\partial v^2}\right|_{st} .$$

Let σ be a plane domain which is the projection of S onto a certain plane and let Γ_o be the projection onto the same plane of the boundary Γ of S. Then

$$(12) \quad \int_S Fe^{ik\phi}dS = \int_\sigma Fe^{ik\phi}H(x,y)d\sigma = -\frac{1}{ik}\left[\frac{2\pi H(x_o,y_o)}{\sqrt{\alpha_1\alpha_3-\alpha_2^2}}F(x_o,y_o)e^{ik\phi(x_o,y_o)} - \right.$$

$$\left. - \int_{\Gamma_o} \frac{FH(x,y)e^{ik\phi}(\vec{m}_o\vec{r}_o)}{(\vec{r}_o \text{grad}\ \phi)} dl \right] + \frac{1}{k}\, O(\tfrac{1}{k}),$$

where $\vec{m}_o$ is the normal to Γ_o, $\vec{r}_1$ is the radius vector on σ with center at the projection of the stationary point, and $H(x,y)$ is the discriminant of the first quadratic form of the surface in the x,y coordinates.

Noting that

$$\sqrt{\alpha_1\alpha_3-\alpha_2^2} = \sqrt{a_1a_3-a_2^2}\left[\frac{\partial(u,v)}{\partial(x,y)}\right],$$

where $\left[\frac{\partial(u,v)}{\partial(x,y)}\right]$ is the Jacobian of the coordinate transformation from x,y to u,v, we find that the quantity

$$\frac{H(x,y)}{\sqrt{\alpha_1\alpha_3-\alpha_2^2}} = \frac{H(u,v)}{\sqrt{a_1a_3-a_2^2}},$$

where $H(u,v)$ is the discriminant of the first quadratic form of the surface in the u,v coordinates, is invariant with respect to any coordinate transformation.

The connection between an element of arc length dl_o of the contour Γ_o and an element of arc length dl of the contour Γ is given by the relation:

$$dl_o = \frac{(\vec{m}\,\vec{r}_o)}{(\vec{m}_o\vec{r}_o)}\,dl,$$

where $\vec{m}$ is normal to the contour Γ.

Thus the representation (12) can be written down as follows:

$$(13)\quad \int_S Fe^{ik\phi}dS = -\frac{1}{ik}\left[\frac{2\pi H(u_o,v_o)}{\sqrt{a_1a_3-a_2^2}}F(u_o,v_o)e^{ik\phi(u_o,v_o)} - \int_\Gamma \frac{Fe^{ik\phi}(\vec{m}\,\vec{r}_o)}{(\vec{r}_o\operatorname{grad}\phi)}dl\right] + \frac{1}{k}O\left(\frac{1}{k}\right).$$

If instead of the function F there is given a vector function $\vec{F}$, then, applying (13) to each of the scalar components of the integral, we obtain the following representation for an integral of the type under consideration:

$$(14)\quad \int_S \vec{F}e^{ik\phi}dS = -\frac{1}{ik}\left[\frac{2\pi H(u_o,v_o)}{\sqrt{a_1a_3-a_2^2}}\vec{F}(u_o,v_o)e^{ik\phi(u_o,v_o)} - \int_\Gamma \frac{\vec{F}e^{ik\phi}(\vec{m}\,\vec{r}_o)}{(\vec{r}_o\operatorname{grad}\phi)}dl\right] + \frac{1}{k}\vec{O}\left(\frac{1}{k}\right).$$

Here the term $\frac{1}{k}\vec{O}(\frac{1}{k})$ stands for the expression:

$$\frac{1}{k}\vec{O}\left(\frac{1}{k}\right) = -\frac{1}{ik}\int_S e^{ik\phi}\left(\nabla, \frac{\vec{r}_o}{(\vec{r}_o\operatorname{grad}\phi)}\right)\vec{F}dS.$$

The above method for obtaining an asymptotic representation for surface integrals [formulas (9)-(14)] is in fact a generalization of the so-called method of stationary phase which is widely applied for approximate computation of one-dimensional integrals of oscillating functions.

3. Derivation of asymptotic formulas for the computation of the electromagnetic field reflected from a surface of arbitrary shape*

The above formulas can be used, in particular, to establish a connection between the laws of geometrical optics and the laws of electrodynamics and to obtain asymptotic expressions for the electromagnetic field reflected from a surface of arbitrary shape in the case of short waves (i.e. for large values of the wave number $k = 2\pi/\lambda$). An analogous problem was first considered by V. A. Fok in 1948. By combining the Fresnel laws of reflection and the laws of ray optics Fok obtained formulas for the computation of the reflected field of an incident plane wave in the approximation of geometrical optics[3]. A generalization of these formulas covering the case of an arbitrary incident wave is found in the paper [2] by V. A. Fok, which is quoted below.

If a diffracting body is an ideal conductor, then the values of the vectors $\vec{E}(p)$ and $\vec{H}(p)$ of the field at a point p in space and the current density $\vec{j}$ induced on the surface of the body by the field of the incident wave $\vec{E}_o$ and $\vec{H}_o$ are connected by the following relations:

$$(15)\qquad \vec{E}(p) = \vec{E}_o(p) - \frac{1}{4\pi\, ik}\sqrt{\frac{\mu}{\varepsilon}}\,\mathrm{rot}_p \int_S \left[\vec{j},\ \mathrm{grad}\,\frac{e^{ikR}}{R}\right] dS,$$

$$(16)\qquad \vec{H}(p) = \vec{H}_o(p) + \frac{1}{4\pi}\int_S \left[\vec{j},\ \mathrm{grad}\,\frac{e^{ikR}}{R}\right] dS.$$

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*The results of this section are due to U. K. Muravév.

For sufficiently short wave lengths the current density $\vec{j}$ is defined up to terms which decrease together with decreasing wave length (increasing k) by the following relations:*

(17) $\vec{j} = 2[\vec{n}\,\vec{H}_o]$ (on the lighted side of the body),

(18) $\vec{j} = 0$ (on the shaded side of the body).

We shall assume that the incident wave $\vec{E}_o$, $\vec{H}_o$ is spherical

$$(19)\quad \vec{E}_o = \overrightarrow{E^o}\,\frac{e^{ikr}}{r}\,; \quad \vec{H}_o = \overrightarrow{H^o}\,\frac{e^{ikr}}{r} \text{ with } \overrightarrow{H^o} = \sqrt{\frac{\varepsilon}{\mu}}\left[\vec{r}_o\overrightarrow{E^o}\right],$$

where $\overrightarrow{E^o}$ and $\overrightarrow{H^o}$ characterize the amplitude and polarization of the incident wave on a unit sphere about the source at 0, and $\vec{r}_o$ is the unit radius vector from 0 to the point in which we define the field $\vec{E}_o$, $\vec{H}_o$.

Using (17), (18), and (19) and ignoring terms of order $\frac{1}{(kR)^2} \ll \frac{1}{kR}$ we obtain in place of (15) and (16) the following expressions:

$$(20)\quad \vec{E}(p) = \vec{E}_o(p) + \frac{ik}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\int\limits_{S_{\text{lighted}}}\left\{\left[\vec{n}\,\overrightarrow{H^o}\right] - \vec{R}_o\left(\vec{R}_o\left[\vec{n}\,\overrightarrow{H^o}\right]\right)\right\}\frac{e^{ik(r+R)}}{rR}\,dS,$$

---

*If the half-shaded part of the body has small curvature, then for the current density in that part of the body we can use the function defined by V.A. Fok[1]. We restrict ourselves to the case when the radii of curvature of the body in the half-shaded part of the body are small in comparison with the wave length and we ignore the boundary effect.

$$\vec{H}(p) = \vec{H}_o(p) + \frac{ik}{2\pi} \int_{S_{\text{lighted}}} \left[\vec{R}_o\left[\vec{n}\,\vec{H^o}\right]\right] \frac{e^{ik(r+R)}}{rR}\, dS. \tag{21}$$

Here $\vec{R}_o$ is a unit radius vector directed from a point on the surface of the body to the observation point p.

For an asymptotic representation of the integrals in (20) and (21) for large values of k we use formula (14).

Let the surface be characterized by Gaussian coordinates u,v and let it be defined by the equation $\vec{v} = \vec{r}(u,v)$. The function $\varnothing$ defined above has, in the integrals under consideration, the following form (notations explained in Fig. 2):

$$\varnothing = r + R = r + \sqrt{r^2 + r_1^2 - 2(\vec{r}\,\vec{r}_1)}.$$

Here r_1 denotes the distance between the source and the observation point ($\vec{r}_1$ is a constant vector) and the condition grad $\varnothing$ [translator: here the condition imposed on grad $\varnothing$ is missing] yields the following system of equations for the determination of the coordinates of the stationary points on the surface of the body:

$$\frac{\partial\varnothing}{\partial u} = \frac{\partial r}{\partial u} + \frac{r\frac{\partial r}{\partial u} - \left(\vec{r}_1\,\frac{\partial\vec{r}}{\partial u}\right)}{R} \tag{22}$$

$$\frac{\partial\varnothing}{\partial v} = \frac{\partial r}{\partial v} + \frac{r\frac{\partial r}{\partial v} - \left(\vec{r}_1\,\frac{\partial\bar{r}}{\partial v}\right)}{R} = 0, \tag{23}$$

which can be written in the form

$$\left(\vec{r}_o - \vec{R}_o, \frac{\partial\vec{r}}{\partial u}\right) = 0,$$

$$\left(\vec{r}_o - \vec{R}_o, \frac{\partial\vec{r}}{\partial v}\right) = 0.$$

The solution of this system of equations yields the following relations between the unit vectors $\vec{R}_o$ and $\vec{r}_o$(cf. Fig. 2) at the stationary point:

(24) $$\vec{R}_o = \vec{r}_o - 2\vec{n}(\vec{n}\,\vec{r}_o),$$

(25) $$\vec{R}_o = \vec{r}_o,$$

where $\vec{n}$ is normal to the surface.

Thus, it is possible to have two types of stationary points:

Type 1: the stationary point is defined by equation (24) which corresponds to the case when, at an observation point belonging to the lighted part of the surface, the angle of incidence is equal to the angle of reflection, and

Type 2: the stationary point is defined by condition (25) and is located at the intersection of the surface with the line connecting the source of the field with the observation point p in the shaded part of the space.

We assume that the surface is such that if p is an observation point belonging to the lighted part of space, then there corresponds to it only one stationary point of the first type; likewise, if p is an observation point belonging to the shaded part of space, then there belongs to it only one stationary point of the second type.

Under these circumstances it is possible, using (14), to write (20) and (21) as follows:*

$$(26) \quad \vec{E}(p) = \vec{E}_0(p) - \sqrt{\frac{\mu}{\varepsilon}}\, \frac{\left[\vec{n}\vec{H}^0\right] - \vec{R}_0\left(\vec{R}_0\left[\vec{n}\,\vec{H}^0\right]\right)}{\sqrt{a_1 a_3 - a_2^2}}\, \frac{e^{ik(r+R)}}{rR}\, H(u,v)\Big|_{st} + \vec{O}\left(\frac{1}{k}\right),$$

$$(27) \quad \vec{H}(p) = \vec{H}_0(p) - \frac{\left[\vec{R}_0\left[\vec{n}\,\vec{H}^0\right]\right]}{\sqrt{a_1 a_3 - a_2^2}}\, \frac{e^{ik(r+R)}}{rR}\, H(u,v)\Big|_{st} + \vec{O}\left(\frac{1}{k}\right).$$

We now express the coefficient $\sqrt{a_1 a_3 - a_2^2}$ in terms of the values of the local parameters of the surface at the stationary point:

$$(28) \quad a_1 = \frac{\partial^2 \phi}{\partial u^2}\Big|_{st} = \frac{\partial^2 r}{\partial u^2} + \frac{\left(\frac{\partial r}{\partial u}\right)^2 + r\,\frac{\partial^2 r}{\partial u^2} - \left(\vec{r}_1\, \frac{\partial^2 \vec{r}}{\partial u^2}\right)}{R} - \frac{\left[r\,\frac{\partial r}{\partial u} - \left(\vec{r}_1\, \frac{\partial \vec{r}}{\partial u}\right)\right]^2}{R^3}\Bigg|_{st}$$

$$(29) \quad a_2 = \frac{\partial^2 \phi}{\partial u \partial v}\Big|_{st} = \frac{\partial^2 r}{\partial u \partial v} + \frac{\left(\frac{\partial r}{\partial u}\,\frac{\partial r}{\partial v}\right) + r\,\frac{\partial^2 r}{\partial u \partial v} - \left(\vec{r}_1\, \frac{\partial^2 \vec{r}}{\partial u \partial v}\right)}{R} -$$

$$- \frac{\left[r\,\frac{\partial r}{\partial u} - \left(\vec{r}_1\, \frac{\partial \vec{r}}{\partial u}\right)\right]\left[r\,\frac{\partial r}{\partial v} - \left(\vec{r}_1\, \frac{\partial \vec{r}}{\partial v}\right)\right]}{R^3}\Bigg|_{st}$$

$$(30) \quad a_3 = \frac{\partial^2 \phi}{\partial v^2}\Big|_{st} = \frac{\partial^2 r}{\partial v^2} + \frac{\left(\frac{\partial r}{\partial v}\right)^2 + r\,\frac{\partial^2 r}{\partial v^2} + \left(\vec{r}_1\, \frac{\partial^2 \vec{r}}{\partial v^2}\right)}{R} - \frac{\left[r\,\frac{\partial r}{\partial v} - \left(\vec{r}_1\, \frac{\partial \vec{r}}{\partial v}\right)\right]^2}{R^3}\Bigg|_{st}$$

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*Here we assume that the function $\phi = v + R$ does not remain constant along the contour which divides the shaded and the lighted portions of the surface of the body, i.e., the sum of the distances from the source to the edge of the body and from the edge to the observation point is not, for a given form of the body, a constant.

For a stationary point of type I the use of condition (24) leads to the following expressions for the coefficients a_1, a_2, and a_3.

$$a_1 = \left(\frac{1}{r} + \frac{1}{R}\right)\left[E - \left(\frac{\partial r}{\partial u}\right)^2\right] + 2\left(\vec{n}\,\vec{r}_o\right) D,$$

$$a_2 = \left(\frac{1}{r} + \frac{1}{R}\right)\left[F - \left(\frac{\partial r}{\partial u}\;\frac{\partial r}{\partial v}\right)\right] + 2\left(\vec{n}\,\vec{r}_o\right) D',$$

$$a_3 = \left(\frac{1}{r} + \frac{1}{R}\right)\left[G - \left(\frac{\partial r}{\partial v}\right)^2\right] + 2\left(\vec{n}\,\vec{r}_o\right) D'',$$

where $E = \left(\frac{\partial \vec{r}}{\partial u}\right)^2$, $F = \left(\frac{\partial \vec{r}}{\partial u}\;\frac{\partial \vec{r}}{\partial v}\right)$, $G = \left(\frac{\partial \vec{r}}{\partial v}\right)^2$ are the coefficients of the I quadratic form of the surface and $D = \left(\vec{n}\;\frac{\partial^2 \vec{r}}{\partial u^2}\right)$, $D' = \left(\vec{n}\,\frac{\partial^2 \vec{r}}{\partial u \partial v}\right)$, $D'' = \left(\vec{n}\,\frac{\partial^2 \vec{r}}{\partial v^2}\right)$ are the coefficients of the II quadratic form of the surface.

Hence

$$\sqrt{a_1 a_3 - a_2^2} = \left(\frac{1}{r} + \frac{1}{R}\right) H \left\{ 1 - \frac{E\left(\frac{\partial r}{\partial v}\right)^2 + G\left(\frac{\partial r}{\partial u}\right)^2 - 2F\left(\frac{\partial r}{\partial u}\;\frac{\partial r}{\partial v}\right)}{H^2} + \right.$$

$$+ \frac{2(\vec{n}\,\vec{r}_o)}{\left(\frac{1}{r} + \frac{1}{R}\right)}\;\frac{ED'' + GD - 2FD'}{H^2} + \frac{4(\vec{n}\,\vec{r}_o)^2}{\left(\frac{1}{r} + \frac{1}{R}\right)^2}\;\frac{DD'' - D'^2}{H^2} -$$

$$\left. - \frac{2(\vec{n}\,\vec{r}_o)}{\left(\frac{1}{r} + \frac{1}{R}\right)}\;\frac{D''\left(\frac{\partial r}{\partial u}\right)^2 + D\left(\frac{\partial r}{\partial v}\right)^2 - 2D'\left(\frac{\partial r}{\partial u}\,\frac{\partial r}{\partial v}\right)}{H^2} \right\}^{1/2} .$$

Here

$$\frac{ED'' + GD - 2FD'}{H^2} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$ is the mean curvature

$$\frac{DD'' + D'^2}{H^2} = \frac{1}{R_1R_2}$$ is the Gaussian curvature.

It is possible to show that the expressions

$$I_1 = \frac{E\left(\frac{\partial r}{\partial v}\right)^2 + G\left(\frac{\partial r}{\partial u}\right)^2 - 2F\left(\frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\right)}{H^2},$$

$$I_2 = \frac{D\left(\frac{\partial r}{\partial v}\right)^2 + D''\left(\frac{\partial r^2}{\partial u}\right) - 2D'\left(\frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\right)}{H^2}$$

are invariant with respect to any transformation of the coordinates of the surface. We transform the coordinates u,v to coordinates ξ, η in such a manner that at the stationary point the curve $\eta = 0$ is the trace of the intersection of the surface with the plane of the vectors $\vec{n}$ and $\vec{r}$ and the curve $\xi = 0$ is orthogonal to the curve $\eta = 0$. Then

$$I_1 = \sin^2\theta; \qquad I_2 = \sin^2\theta\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_0}\right),$$

where θ is the angle of incidence and R_0 is the radius of curvature of the normal section of the surface by means of the plane of incidence.

We thus have for the coefficient in question:

$$\sqrt{a_1a_3-a_2^2} = \left(\frac{1}{r} + \frac{1}{R}\right)H\left\{\cos^2\theta - \frac{2r\,R\cos\theta}{r + R}\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\cos^2\theta + \frac{\sin^2\theta}{R_0}\right] + \right.$$

$$\left. + \frac{4r^2R^2\cos^2\theta}{(r + R)^2}\frac{1}{R_1R_2}\right\}^{1/2}.$$

In the case of a stationary point of type II we can, using condition (25), reduce the expressions (28)-(30) to the form:

$$a_1 = \left(\frac{1}{r} + \frac{1}{R}\right)\left[E - \left(\frac{\partial r}{\partial u}\right)^2\right],$$

$$a_2 = \left(\frac{1}{r} + \frac{1}{R}\right)\left[F - \left(\frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\right)\right],$$

$$a_3 = \left(\frac{1}{r} + \frac{1}{R}\right)\left[G - \left(\frac{\partial r}{\partial v}\right)^2\right],$$

whence

$$(32) \qquad \sqrt{a_1 a_3 - a_2^2} = \left(\frac{1}{r} + \frac{1}{R}\right) H \cos \theta .$$

Thus, using relations (24), (26), (27), and (31), we can represent the field in the lighted part of space as follows:

$$(33) \qquad \vec{E}(p) = \vec{E}_o(p) - \left[\overrightarrow{E^o} - 2\vec{n}(\vec{n}\,\overrightarrow{E^o})\right]_{st} B\,\frac{e^{ik(r+R)}}{r+R} + \vec{O}\left(\frac{1}{k}\right),$$

$$(34) \qquad \vec{H}(p) = \vec{H}_o(p) + \left[\overrightarrow{H^o} - 2\vec{n}(\vec{n}\,\overrightarrow{H^o})\right]_{st} B\,\frac{e^{ik(r+R)}}{r+R} + \vec{O}\left(\frac{1}{k}\right),$$

where

$$(35) \qquad B = \left\{1 - \frac{2rR\cos\theta}{r+R}\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{tg^2\theta}{R_o}\right] + \frac{4r^2R^2}{(r+R)^2}\,\frac{1}{R_1R_2}\right\}^{-1/2} .$$

The second terms in (33) and (34) determine the amplitude, polarization, and phase of the reflected field in the approximation of geometrical optics.

Using condition (25) and the corresponding formula (32) in computing (26) and (27) we can show that

$$-\sqrt{\frac{\mu}{\varepsilon}}\,\frac{[n\,\vec{H}^{o}] - \vec{R}_o(\vec{R}_o[\vec{n}\,\vec{H}_o])}{\sqrt{a_1a_3-a_2^2}}\;\frac{e^{ik(r+R)}}{rR}\,H\Bigg|_{st} = -\vec{E}_o(p)$$

$$-\frac{[\vec{R}_o[\vec{n}\,\vec{H}^{o}]]}{\sqrt{a_1a_3-a_2^2}}\;\frac{e^{ik(r+R)}}{rR}\,H\Bigg|_{st} = -\vec{H}_o(p),$$

i.e., in the approximation of geometrical optics the field in the shaded part of space is equal to zero.

In the above considerations we assumed that the function $\phi = r + R$ is not constant along the boundary of the lighted and shaded domains. If $r + R = \text{const.}$ on Γ (this happens, for instance, if the observation point and the source are on the axis of the optical system which is a mirror or a lens whose form is that of a solid of revolution) then we have the following representations for the field:

For the lighted part of space

$$\vec{E}(p) = \vec{E}_o(p) + \vec{E}_{geom.} + \vec{E}_{diff.} + \vec{O}(\tfrac{1}{k});$$

for the shaded part of space

$$\vec{E}(p) = \vec{E}_{diff.} + \vec{O}\,(\tfrac{1}{k}),$$

where $\vec{E}_{geom.}$ is defined by (33) and

$$\vec{E}_{diff.} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\;e^{ik(r+R)\Gamma}\int_{\Gamma}\frac{[\vec{n}\,\vec{H}^{o}] - \vec{R}_o(\vec{R}_o[\vec{n}\,\vec{H}^{o}])}{rR}\;\frac{\frac{\partial(r+R)}{\partial m}}{[\text{grad}\,(r+R)]^2}\,dl.$$

This shows that in this case diffraction phenomena will take place for arbitrarily short wave lengths and $\vec{E}_o$, $\vec{E}_{geom.}$, and $\vec{E}_{diff.}$ are all of the same order of magnitude.

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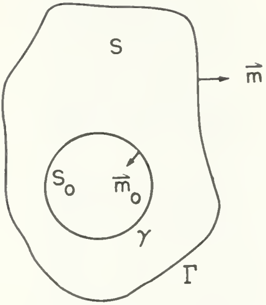

Figure 1

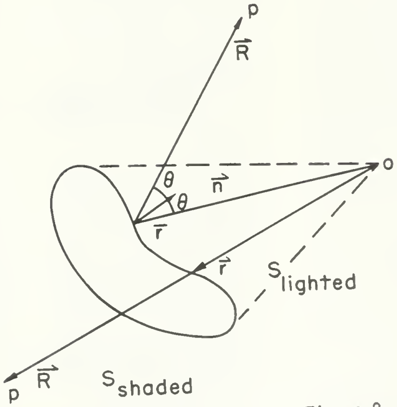

Figure 2

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